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Improved Gain-Scheduled Control Design for Rational Nonlinear Discrete-Time Systems with Input Saturation*

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***Abstract*— An improved method of control design for rational nonlinear discrete-time systems with input saturation is proposed in this paper. Using Difference-Algebraic Representation (DAR) and parameter-dependent Lyapunov functions, a novel regional stabilization condition in terms of Linear Matrix Inequalities (LMI) is presented. Two optimization problems are addressed to either obtain the largest estimated Domain of Attraction (DoA) or minimize the ℓ_2-gain from the energy-bounded disturbance input to the performance output. Numerical examples illustrate the potential of the proposed approach.**

I. INTRODUCTION

The predominance of nonlinear systems is widely known in the context of control systems [1]. Several real-life applications, such as electromechanical, electronic, chemical, and biological systems, present nonlinear behavior. Considering nonlinear characteristics of certain systems is essential to model nonlinear phenomena and ensure the validity of results. Besides that, nonlinear control strategies can be important to achieve better performance for the closed-loop control system. However, stability analysis and control design for nonlinear systems are very challenging.

The derivation of generic stability analysis and control synthesis methods for nonlinear systems is a difficult task due to the diversity of nonlinear phenomena and so most methodologies concern well-defined classes of systems. Moreover, for a nonlinear system is not always possible to ensure global stability. Therefore, an approach widely explored by researchers is to consider a compact region in the state space in which an estimated Domain of Attraction (DoA) can be determined [1]. These investigations have considered state constraints generated by limitations in the physical system or associated with the domain of validity of the system's mathematical model. Recently, control input constraints arising from actuator saturation have also been studied more deeply [2], [3].

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Within this context, this work aims to develop improved stabilization conditions for discrete-time nonlinear parameter-varying systems subject to input saturation. In this research, we consider the class of *rational* systems, which represents a wide range of phenomena in practical applications [4]. A particular way of expressing this class of systems is by using the well-known Difference-Algebraic Representation (DAR). From a DAR, it is possible to obtain an exact representation of rational systems in discrete-time as a set of algebraic and difference equations. Thus, it is possible to apply the well-established Lyapunov theory and Linear Matrix Inequality (LMI)-based tools, which have given rise to a large number of works on the stability analysis [5]–[9] and control design [10]–[14] for systems in DAR form.

Regarding discrete-time nonlinear systems with input saturation, due to inherent difficulties in developing synthesis conditions, DAR-based approaches have used quadratic Lyapunov functions and linear state feedback controllers to investigate stabilization and DoA estimation [13]. However, improvements were recently reported in [14] based on parameter-dependent Lyapunov functions and gain-scheduled control. The purpose of our paper is to present further developments upon these previous works by incorporating information about the system's nonlinearities in the control law.

In this sense, this paper provides novel stabilization conditions to design *nonlinear* gain-scheduled state feedback controllers for parameter-varying nonlinear discrete-time systems with input saturation described in a DAR form. More specifically, our main contributions can be summarized as follows:

- A new sufficient condition for regional stabilization of discrete-time nonlinear systems with input saturation is provided. Input saturation is incorporated to the LMI conditions by using a polytopic description as proposed in [2]. The novel set of LMIs is also obtained by considering parameter-dependent Lyapunov functions, and no iterative algorithms are required.
- The proposed condition is extended to cope with two main problems of control theory related to: (*i*) minimizing an upper-bound to the induced ℓ_2-gain from an energy-bounded disturbance input to the performance output, considering zero initial conditions; and (*ii*) obtaining the largest estimated Domain of Attraction (DoA) for the closed-loop system, in the absence of disturbances.
- In contrast to existing methodologies for DAR systems

in a similar context [13], [14], the proposed approach is more general and offers additional degrees of freedom in the control design, providing less conservative results.

Outline: Section II presents the problem formulation. Conditions to synthesize nonlinear gain-scheduled controllers are given in Section III. Section IV brings some numerical examples. The concluding remarks are provided in Section V.

Notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices, I_n is the $n \times n$ identity matrix and diag$\{\dots\}$ stands for a block-diagonal matrix. For a real matrix M, $M^\top$ denotes its transpose, $M > 0$ ($M \geq 0$) means that M is symmetric positive definite (semi-positive definite) and $M_{(i)}$ is the i-th row. He$\{M\}$ denotes the symmetric term $M + M^\top$. For a symmetric block matrix, the symbol $\star$ stands for the transpose of the blocks outside the main diagonal block. Let $\mathcal{I}_n = [1,\ n] \subset \mathbb{N},\ n \in \mathbb{N}$. For two sets $\mathcal{X} \subset \mathbb{R}^{n_x}$ and $\Delta \subset \mathbb{R}^{n_\delta}$, the notation $\mathcal{X} \times \Delta \subset \mathbb{R}^{n_x+n_\delta}$ is the Cartesian product of $\mathcal{X}$ and Δ. $\Lambda_1 := \left\{\alpha_{\mathrm{p},k} \in \mathbb{R}^N : \sum_{\mathrm{v}=1}^{N} \alpha_{\mathrm{p}_{(\mathrm{v})k}} = 1,\ \alpha_{\mathrm{p}_{(\mathrm{v})k}} \geq 0\right\}$ represents the unitary simplex, where p represents an index used to distinguish different polytopes, N is the number of vertices and $\alpha_{\mathrm{p}_{(\mathrm{v})k}}$ is the vth entry in the vector at time k. Matrices of affine functions of (x_k, δ_k) are given by: $M(x_k, \delta_k) = \sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}} M_{il}$, where N_x and N_δ denote the number of vertices of the x_k and δ_k polytopes, respectively. The norm in the ℓ_2-space of summable sequences is defined as: $\|z_k\|_2 = \left(\sum_{i=1}^{\infty} |z_{(i)k}|^2\right)^{1/2} < \infty$.

II. PROBLEM FORMULATION

Consider the following class of discrete-time nonlinear systems:

$$\begin{aligned} x_{k+1} &= f(x_k, \delta_k) + g(x_k, \delta_k)\mathrm{sat}(u_k) + h(x_k, \delta_k)w_k, \\ z_k &= f_z(x_k, \delta_k) + g_z(x_k, \delta_k)\mathrm{sat}(u_k) + h_z(x_k, \delta_k)w_k, \end{aligned} \quad (1)$$

where $x_k \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is the state vector of the system, $z_k \in \mathbb{R}^{n_z}$ is the performance output, $w_k \in \mathbb{R}^{n_w}$ is the exogenous disturbance input which is supposed to be an arbitrary signal in the ℓ_2 space, $\delta_k \in \Delta \subseteq \mathbb{R}^{n_\delta}$ is a time-varying parameter vector supposed to be known, $u_k = \begin{bmatrix} u_{(1)k} & u_{(2)k} & \cdots & u_{(n_u)k} \end{bmatrix} \in \mathbb{R}^{n_u}$ is the control input, and the saturation function $\mathrm{sat}(u_k) = \begin{bmatrix} \mathrm{sat}(u_{(1)k}) & \mathrm{sat}(u_{(2)k}) & \cdots & \mathrm{sat}(u_{(n_u)k}) \end{bmatrix} \in \mathbb{R}^{n_u}$ corresponds to:

$$\mathrm{sat}\left(u_{(s)k}\right) := \mathrm{sign}\left(u_{(s)k}\right) \times \min\left\{\left|u_{(s)k}\right|, u_{0(s)}\right\},\ s \in \mathcal{I}_{n_u},$$

such that $u_{0(s)}$ is the maximum absolute value of $u_{(s)k}$.

The following assumptions are considered for system (1).

Assumption 1: Functions $f(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_x}$ (with $f(0, \delta_k) = 0$), $f_z(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_z}$, (with $f_z(0, \delta_k) = 0$), $g(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_x \times n_u}$, $g_z(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_z \times n_u}$ are rational functions, $h(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_x \times n_w}$ and $h_z(\cdot) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\delta} \to \mathbb{R}^{n_z \times n_w}$ are affine functions, well-posed on $\mathcal{X} \times \Delta$.

Assumption 2: The disturbance input vector w_k lies inside the following class of square summable sequences:

$$\mathcal{W} := \left\{w_k \in \mathbb{R}^{n_w} : \|w_k\|_2^2 \leq \lambda^{-1}\right\},\ \text{for some } \lambda > 0. \quad (2)$$

Assumption 1 regards the class of *rational systems* and guarantees existence of the solutions of the difference equation in a neighborhood $\mathcal{X} \times \Delta$ of the equilibrium point $f(0, \delta_k) = 0,\ \forall \delta_k \in \Delta$.

In [5], it is shown that rational systems can be recast as a DAR given in (3) (on the top of Page 3), where $\pi_k := \pi(x_k, \delta_k, \mathrm{sat}(u_k)) \in \mathbb{R}^{n_\pi}$ is an auxiliary vector of nonlinear functions with respect to (x_k, δ_k) and linear on $(\mathrm{sat}(u_k))$. All system matrices are affine functions of (x_k, δ_k) with appropriate dimensions, such that $\Omega_2(x_k, \delta_k)$ is a square full-rank matrix for all $(x_k, \delta_k) \in \mathcal{X} \times \Delta$.

The decomposition of the nonlinear system in a DAR form is not unique (see [6]). On the other hand, the correctness of a DAR can be verified by replacing the nonlinearity vector π_k given by the null algebraic equation in (3) with the corresponding expression below so that (1) is obtained.

$$\pi_k = -\Omega_2^{-1}(x_k, \delta_k)\left[\Omega_1(x_k, \delta_k)x_k + \Omega_3(x_k, \delta_k)\mathrm{sat}(u_k)\right].$$

In practical situations where physical limitations or the region of validity of the system's model are important aspects, one must consider a restricted domain for the system's state variables excursion. In this investigation, the state trajectories of system (3) will be constrained into the following polyhedral set:

$$\mathcal{X} := \left\{x_k \in \mathbb{R}^{n_x} : a_p^\top x_k \leq 1, \quad p \in \mathcal{I}_{n_e}\right\},$$

where a_p is a constant n_x-dimensional vector of parameters, and n_e is the number of hyperplanes which characterizes the region $\mathcal{X}$.

For the stabilization of the DAR model (3), the following nonlinear control law is proposed:

$$u_k = K(x_k, \delta_k)G^{-1}(x_k, \delta_k)x_k + R(x_k, \delta_k)N^{-1}(x_k, \delta_k)\pi_k, \quad (4)$$

with $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}, G(x_k, \delta_k) \in \mathbb{R}^{n_x \times n_x}$, $R(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ and $N(x_k, \delta_k) \in \mathbb{R}^{n_\pi \times n_\pi}$ matrices of affine functions with respect to (x_k, δ_k), to be determined.

Remark 1: Note that the proposed control law includes the particular case $u_k = K(x_k, \delta_k)G^{-1}(x_k, \delta_k)x_k$, discussed in [14], by considering $R(x_k, \delta_k) = 0$.

Following the work in [2], the saturation vector function can be represented by the polytopic description stated in Lemma 1 in the Appendix. For this purpose, consider the auxiliary vector

$$v_k = H(x_k, \delta_k)G^{-1}(x_k, \delta_k)x_k + S(x_k, \delta_k)N^{-1}(x_k, \delta_k)\pi_k, \quad (5)$$

with $H(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$ and $S(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ matrices of affine functions with respect to (x_k, δ_k) to be determined. Replacing (4) and (5) in (19), we have

$$\begin{aligned} \mathrm{sat}(u_k) = &\left[D(\alpha_{d_{(r)k}})K(x_k, \delta_k)G^{-1}(x_k, \delta_k)\right. \\ &\left. + D^-(\alpha_{d_{(r)k}})H(x_k, \delta_k)G^{-1}(x_k, \delta_k)\right] x_k \\ &+ \left[D(\alpha_{d_{(r)k}})R(x_k, \delta_k)N^{-1}(x_k, \delta_k)\right. \\ &\left. + D^-(\alpha_{d_{(r)k}})S(x_k, \delta_k)N^{-1}(x_k, \delta_k)\right] \pi_k, \end{aligned} \quad (6)$$

where $D(\alpha_{d_{(r)k}}) = \sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r,\ D^-(\alpha_{d_{(r)k}}) = \sum_{r=1}^{N_u} \alpha_{d_{(r)k}} D_r^-$, and $\alpha_{d_{(r)k}} \in \Lambda_1$.

$$\begin{aligned} x_{k+1} &= A_1(x_k,\delta_k)x_k + A_2(x_k,\delta_k)\pi_k + A_3(x_k,\delta_k)\operatorname{sat}(u_k) + A_4(x_k,\delta_k)w_k,\\ z_k &= C_1(x_k,\delta_k)x_k + C_2(x_k,\delta_k)\pi_k + C_3(x_k,\delta_k)\operatorname{sat}(u_k) + C_4(x_k,\delta_k)w_k,\\ 0 &= \Omega_1(x_k,\delta_k)x_k + \Omega_2(x_k,\delta_k)\pi_k + \Omega_3(x_k,\delta_k)\operatorname{sat}(u_k). \end{aligned} \tag{3}$$

Replacing (6) in (3), the closed-loop system is given by

$$\begin{aligned} x_{k+1} &= \mathbf{A_{1cl}}x_k + \mathbf{A_{2cl}}\pi_k + A_4(x_k,\delta_k)w_k,\\ 0 &= \mathbf{\Omega_{1cl}}x_k + \mathbf{\Omega_{2cl}}\pi_k,\\ z_k &= \mathbf{C_{1cl}}x_k + \mathbf{C_{2cl}}\pi_k + C_4(x_k,\delta_k)w_k, \end{aligned} \tag{7}$$

such that

$$\begin{aligned} \mathbf{M_{1cl}} &= M_1(x_k,\delta_k) + M_3(x_k,\delta_k)\Theta_1 G^{-1}(x_k,\delta_k),\\ \mathbf{M_{2cl}} &= M_2(x_k,\delta_k) + M_3(x_k,\delta_k)\Theta_2 N^{-1}(x_k,\delta_k), \end{aligned}$$

where $\mathbf{M_{icl}}$ and M_i are placeholders for the matrices $\mathbf{A_{icl}}$, $\mathbf{\Omega_{icl}}$, $\mathbf{C_{icl}}$, and, A_i, Ω_i, C_i, $i \in \mathcal{I}_2$, respectively.

$$\begin{aligned} \Theta_1 &= \left[D(\alpha_{d_{(r)k}})K(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})H(x_k,\delta_k)\right],\\ \Theta_2 &= \left[D(\alpha_{d_{(r)k}})R(x_k,\delta_k) + D^-(\alpha_{d_{(r)k}})S(x_k,\delta_k)\right]. \end{aligned}$$

To investigate the regional stabilization of system (1), consider the following Lyapunov function candidate:

$$V(x_k,\delta_k) = x_k^\top P^{-1}(\delta_k)x_k, \tag{8}$$

where $P(\delta_k) = \sum_{l=1}^{N_\delta}\alpha_{\delta_{(l)k}}P_l,\ P_l = P_l^\top > 0$.

Remark 2: The level set associated with (8) is defined by

$$\mathcal{L}_\mathcal{V} := \left\{x_k \in \mathbb{R}^{n_x} : V(x_k,\delta_k) \le \lambda^{-1}, \quad \forall \delta_k \in \Delta\right\},$$

where λ is the positive scalar used in (2).

An alternative to estimating the level set $\mathcal{L}_\mathcal{V}$ is to consider the following subset:

$$\mathcal{E}_\mathcal{V} = \bigcap_{l\in\{1,\ldots N_\delta\}} \mathcal{E}\left(P_l^{-1},\lambda^{-1}\right) \subseteq \mathcal{L}_\mathcal{V}, \tag{9}$$

with $\mathcal{E}\left(P_l^{-1},\lambda^{-1}\right) := \left\{x_k \in \mathbb{R}^{n_x} : x_k^T P_l^{-1}x_k \le \lambda^{-1}\right\}$.

If $\Delta V = V(x_{k+1},\delta_{k+1}) - V(x_k,\delta_k) < 0$, $\forall x_k \in \mathcal{L}_\mathcal{V}$, then (8) is said to be a Lyapunov function and $\mathcal{L}_\mathcal{V}$ is an invariant set with respect to the closed-loop system (7). Further, if $x_0 \in \mathcal{L}_\mathcal{V}$ and $w_k = 0$, $\forall k \ge 0$, then $x_k \to 0$, when $k \to \infty$. Thus, $\mathcal{L}_\mathcal{V}$ is a subset of the DoA of the closed-loop system (7).

On the other hand, by considering $x_0 = 0$ and $w_k \in \mathcal{W}$, an upper-bound γ for the ℓ_2-performance from the disturbance w_k to the output z_k satisfies $\|z_k\|_2 \le \gamma\|w_k\|_2$. In this case, from Lemma 2 in the Appendix, it is possible to obtain synthesis conditions relating the system stabilization region to the admissible energy-bounded disturbance.

In light of the previous discussions, this research is particularly concerned with the following control problems.

Problem 1 (Input-to-output performance): Design a controller (4) for system (1) that minimizes an upper-bound γ for the ℓ_2-gain from the disturbance w_k to the performance output z_k, for $x_0 = 0$, and also ensures that system states x_k remain bounded in $\mathcal{L}_\mathcal{V}$ for all $k \ge 0$. Moreover, if there exists $\bar{k} > 0$ such that $w_k = 0$, $\forall k \ge \bar{k}$, then $x_k \to 0$, when $k \to \infty$.

Problem 2 (DoA estimation): Consider system (1), for $x_0 \in \mathcal{L}_V$ and $w_k = 0$, $\forall k \ge 0$. Design a controller as in (4) such that $\mathcal{L}_V \subset \mathcal{X}$, $\forall \delta_k \in \Delta$ is as large as possible, and $\mathcal{L}_V$ is a positively invariant set for the closed-loop system (7).

III. MAIN RESULTS

A novel sufficient condition to synthesize the gain-scheduled controller (4) to stabilize the nonlinear system (1) with a guaranteed upper bound γ for the induced ℓ_2-gain from $w_k \in \mathcal{W}$ to z_k is stated in the following theorem.

Theorem 1: Consider system (1), with $w_k \in \mathcal{W}$ for a given scalar $\lambda > 0$. If there exist a positive scalar μ, a symmetric matrix $P(\delta_k) \in \mathbb{R}^{n_x\times n_x}$, and matrices $G(x_k,\delta_k) \in \mathbb{R}^{n_x\times n_x}$, $K(x_k,\delta_k) \in \mathbb{R}^{n_u\times n_x}$, $H(x_k,\delta_k) \in \mathbb{R}^{n_u\times n_x}$, $R(x_k,\delta_k) \in \mathbb{R}^{n_u\times n_\pi}$, $S(x_k,\delta_k) \in \mathbb{R}^{n_u\times n_\pi}$ and $N(x_k,\delta_k) \in \mathbb{R}^{n_\pi\times n_\pi}$, satisfying the following inequalities, $\forall x_k \in \mathcal{X}, \forall \delta_k \in \Delta$, $\forall p \in \mathcal{I}_{n_e}$, and $\forall s \in \mathcal{I}_{n_u}$:

$$\begin{bmatrix} \mathcal{A}_{11}^1 & \star & \star & \star & \star\\ \mathcal{A}_{21}^1 & -P(\delta_{k+1}) & \star & \star & \star\\ \mathcal{A}_{31}^1 & \mathcal{A}_{32}^1 & \mathcal{A}_{33}^1 & \star & \star\\ 0 & A_4^\top(x_k,\delta_k) & 0 & -I_{n_w} & \star\\ \mathcal{A}_{51}^1 & 0 & \mathcal{A}_{53}^1 & C_4(x_k,\delta_k) & -\mu I_{n_z} \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} \lambda u_{0(s)}^2 & \star & \star\\ H_{(s)}^\top(x_k,\delta_k) & \mathrm{He}\{G(x_k,\delta_k)\} - P(\delta_k) & \star\\ S_{(s)}^\top(x_k,\delta_k) & \mathcal{B}_{32}^1 & \mathcal{B}_{33}^1 \end{bmatrix} \ge 0, \tag{11}$$

$$\begin{bmatrix} \lambda & \star\\ G^\top(x_k,\delta_k)a_p & \mathrm{He}\{G(x_k,\delta_k)\} - P(\delta_k) \end{bmatrix} \ge 0, \tag{12}$$

where

$$\begin{aligned} \mathcal{A}_{11}^1 &= -\mathrm{He}\{G(x_k,\delta_k)\} + P(\delta_k),\\ \mathcal{A}_{21}^1 &= A_1(x_k,\delta_k)G(x_k,\delta_k) + A_3(x_k,\delta_k)\Theta_1,\\ \mathcal{A}_{31}^1 &= \Omega_1(x_k,\delta_k)G(x_k,\delta_k) + \Omega_3(x_k,\delta_k)\Theta_1,\\ \mathcal{A}_{51}^1 &= C_1(x_k,\delta_k)G(x_k,\delta_k) + C_3(x_k,\delta_k)\Theta_1,\\ \mathcal{A}_{32}^1 &= N^\top(x_k,\delta_k)A_2^\top(x_k,\delta_k) + \Theta_2^\top A_3^\top(x_k,\delta_k),\\ \mathcal{A}_{33}^1 &= \mathrm{He}\{\Omega_2(x_k,\delta_k)N(x_k,\delta_k) + \Omega_3(x_k,\delta_k)\Theta_2\},\\ \mathcal{A}_{53}^1 &= C_2(x_k,\delta_k)N(x_k,\delta_k) + C_3(x_k,\delta_k)\Theta_2,\\ \mathcal{B}_{32}^1 &= -\Omega_1(x_k,\delta_k)G(x_k,\delta_k) - \Omega_3(x_k,\delta_k)\Theta_1,\\ \mathcal{B}_{33}^1 &= -\mathrm{He}\{\Omega_2(x_k,\delta_k)N(x_k,\delta_k) + \Omega_3(x_k,\delta_k)\Theta_2\}, \end{aligned}$$

then there exist a Lyapunov function (8) and a controller (4) such that, for zero initial conditions ($x_0 = 0$), x_k remains bounded in $\mathcal{L}_\mathcal{V}$ and $\|z\|_2 \le \gamma\|w\|_2$, $\forall w_k \in \mathcal{W}$, with $\gamma = \sqrt{\mu}$. Moreover, if there exists $\bar{k} > 0$ such that $w_k = 0$, $\forall k \ge \bar{k}$, then $x_k \to 0$, as $k \to \infty$.

Proof: By using the property that

$$G^\top P^{-1}G \ge \mathrm{He}\{G\} - P, \tag{13}$$

if inequality (10) holds, it will be satisfied for $\mathcal{A}_{11}^1 = -G^\top(x_k, \delta_k)P^{-1}(\delta_k)G(x_k, \delta_k)$.

From the feasibility of $\mathcal{A}_{11}^1 < 0$ and $\mathcal{A}_{33}^1 < 0$, one can infer that $G(x_k, \delta_k)$ and $N(x_k, \delta_k)$ must be invertible. Thus, one can apply a congruence transformation pre- and post-multiplying inequality (10) by $\text{diag}\{G^{-T}(x_k, \delta_k), P^{-1}(\delta_{k+1}), N^{-T}(x_k, \delta_k), I_{n_w}, I_{n_z}\}$ and its transpose, respectively, to obtain

$$\begin{bmatrix} \mathcal{A}_{11}^2 & \star & \star & \star & \star \\ \mathcal{A}_{21}^2 & -P^{-1}(\delta_{k+1}) & \star & \star & \star \\ \mathcal{A}_{31}^2 & \mathcal{A}_{32}^2 & \mathcal{A}_{33}^2 & \star & \star \\ 0 & \mathcal{A}_{42}^2 & 0 & -I_{n_w} & \star \\ \mathcal{A}_{51}^2 & 0 & \mathcal{A}_{53}^2 & C_4(x_k, \delta_k) & -\mu I_{n_z} \end{bmatrix} < 0,$$

$$\begin{aligned} &\mathcal{A}_{11}^2 = -P^{-1}(\delta_k), && \mathcal{A}_{33}^2 = \text{He}\left\{N^{-T}(x_k, \delta_k)\mathbf{\Omega_{2cl}}\right\} \\ &\mathcal{A}_{21}^2 = P^{-1}(\delta_{k+1})\mathbf{A_{1cl}}, && \mathcal{A}_{42}^2 = A_4^\top(x_k, \delta_k)P^{-1}(\delta_{k+1}) \\ &\mathcal{A}_{31}^2 = N^{-T}(x_k, \delta_k)\mathbf{\Omega_{1cl}}, && \mathcal{A}_{51}^2 = \mathbf{C_{1cl}} \\ &\mathcal{A}_{32}^2 = \mathbf{A_{2cl}^\top} P^{-1}(\delta_{k+1}), && \mathcal{A}_{53}^2 = \mathbf{C_{2cl}}. \end{aligned}$$

Applying the Schur complement and choosing $\mu = \gamma^2$, the above inequality can be recast as:

$$\Xi_1 + L\Xi_2 + \Xi_2^\top L^\top + \gamma^{-2}\Xi_3^\top \Xi_3 < 0, \tag{14}$$

where

$$\begin{aligned} \Xi_1 &= \text{diag}\left\{-P^{-1}(\delta_k), P^{-1}(\delta_{k+1}), 0_{n_\pi}, -I_{n_w}\right\}, \\ L &= \begin{bmatrix} 0 & P^{-1}(\delta_{k+1}) & 0 & 0 \\ 0 & 0 & N^{-1}(x_k, \delta_k) & 0 \end{bmatrix}^\top, \\ \Xi_2 &= \begin{bmatrix} \mathbf{A_{1cl}} & -I_{n_x} & \mathbf{A_{2cl}} & A_4(x_k, \delta_k) \\ \mathbf{\Omega_{1cl}} & 0 & \mathbf{\Omega_{2cl}} & 0 \end{bmatrix}, \\ \Xi_3 &= \begin{bmatrix} \mathbf{C_{1cl}} & 0 & \mathbf{C_{2cl}} & C_4(x_k, \delta_k) \end{bmatrix}. \end{aligned}$$

Pre- and post-multiplying (14) by $\begin{bmatrix} x_k^\top & x_{k+1}^\top & \pi_k^\top & w_k^\top \end{bmatrix}$ and its transpose results in (20), in Lemma 2. This proves that if the condition (10) is feasible, then $V(x_k, \delta_k)$ is a Lyapunov function and the controller (4) ensures that for zero initial conditions, the origin of the closed-loop system is input-to-output locally stable with an upper bound γ on the ℓ_2-gain from w_k to z_k, $\forall x_k \in \mathcal{X}, \forall \delta_k \in \Delta$ and $w_k \in \mathcal{W}$.

Using the property (13) and multiplying inequality (11) by $\text{diag}\{1, G^{-T}(x_k, \delta_k), N^{-T}(x_k, \delta_k)\}$ on the left and its transpose on the right, followed by the well-known Schur complement, we obtain

$$\Upsilon^\top \sigma \Upsilon + \begin{bmatrix} -P^{-1}(\delta_k) & \mathbf{\Omega_{1cl}^\top} N^{-1}(x_k, \delta_k) \\ \star & \text{He}\left\{N^{-T}(x_k, \delta_k)\mathbf{\Omega_{2cl}}\right\} \end{bmatrix} \leq 0,$$

with $\sigma = 1/\lambda u_{0(s)}^2$ and

$$\Upsilon = \begin{bmatrix} H_{(s)}(x_k, \delta_k)G^{-1}(x_k, \delta_k) & S_{(s)}(x_k, \delta_k)N^{-1}(x_k, \delta_k) \end{bmatrix}.$$

Pre- and post-multiplying the above, respectively, by $\begin{bmatrix} x_k^\top & \pi_k^\top \end{bmatrix}$ and its transpose, lead to:

$$v_{(s)k}^\top (\lambda u_{0(s)}^2)^{-1} v_{(s)k} - x_k^\top P^{-1}(\delta_k)x_k \leq 0.$$

Considering the S-procedure we have $v_{(s)k}^\top v_{(s)k} \leq u_{0(s)}^2$, $\forall (x_k, \delta_k)$, such that $x_k^\top P^{-1}(\delta_k)x_k \leq \lambda^{-1}$, or $|v_{(s)k}| \leq u_{0(s)}$, $\forall x_k \in \mathcal{L}_\mathcal{V}$. Thus, Lemma 1 is satisfied for all $x_k \in \mathcal{L}_\mathcal{V}$.

Using again the property (13), multiplying (12) by $\text{diag}\{1, G^{-T}(x_k, \delta_k)\}$ on the left and its transpose on the right, and applying the Schur complement we have $a_p \lambda^{-1} a_p^\top - P^{-1}(\delta_k) \leq 0$.

Then, multiplying the last inequality by $x_k^\top$ on the left and x_k on the right and considering the S-procedure leads to $x_k^\top a_p a_p^\top x_k \leq 1$, $\forall x_k : x_k^\top P^{-1}(\delta_k)x_k \leq \lambda^{-1}$. Thus, $|a_p^\top x_k| \leq 1$, $\forall p \in \mathcal{I}_{n_e}$, $\forall x_k \in \mathcal{L}_\mathcal{V}$. This proves the inclusion $\mathcal{L}_\mathcal{V} \in \mathcal{X}$, which concludes the proof. ∎

Notice that in Theorem 1, inequalities are polynomially dependent on $(x_k, \delta_k, \delta_{k+1})$, so Lemma 3 in the Appendix is used to have LMI relaxations employed to convert those conditions into finite sets of LMIs. In [14], one can find a guided proof of Lemma 3.

In the next Corollary, Theorem 1 can be used to solve Problem 1 as described previously.

Corollary 1: For a given disturbance energy level λ^{-1}, the upper-bound γ for the ℓ_2-gain from w_k to z_k can be minimized by solving the following optimization problem for all $\delta_k \in \Delta$ and $x_k \in \mathcal{X}$:

$$\min\ \mu \quad \text{subject to (10)} - \text{(12)}. \tag{15}$$

Regarding Problem 2, an alternative to find the largest DoA is to consider the following subset of $\mathcal{L}_V$

$$\mathcal{E}\left(Q^{-1}, 1\right) \subseteq \bigcap_{l \in \{1, \dots N_\delta\}} \mathcal{E}\left(P_l^{-1}, 1\right).$$

In the next Corollary, Theorem 1 can also be adapted to solve Problem 2.

Corollary 2: Consider system (1), with the disturbance input $w_k = 0$. Disregard the influence of the disturbance input by removing the 4th and 5th lines and columns of the matrix in (10) and consider $\lambda = 1$ in (11) and (12). If there exist symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, and matrices $G(x_k, \delta_k) \in \mathbb{R}^{n_x \times n_x}$, $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $H(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$, $R(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$, $S(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ and $N(x_k, \delta_k) \in \mathbb{R}^{n_\pi \times n_\pi}$, satisfying the following optimization problem for all $x_k \in \mathcal{X}, \delta_k \in \Delta$, $p \in \mathcal{I}_{n_e}$, and $s \in \mathcal{I}_{n_u}$:

$$\begin{cases} \max\left(\log\det(Q)\right) \\ \text{subject to (10)} - \text{(12), and } Q - P_l > 0, \end{cases} \tag{16}$$

then the controller (4) asymptotically stabilizes the closed-loop system (7) and $\mathcal{E}_V$ is an estimate of the DoA.

Proof: The additional inequality in (16) ensures that $\mathcal{E}\left(Q^{-1}, 1\right) \subseteq \mathcal{E}_V$, in (9). The rest of the proof follows in a straightforward way as in the proof of Theorem 1. ∎

IV. NUMERICAL EXAMPLES

In this section, numerical examples are presented to demonstrate the effectiveness of the proposed methodology. The tests were implemented using MATLAB (R2019), the parser Yalmip and the solver Mosek.

Example 1: Consider the following nonlinear system (without time-varying parameter) borrowed from [13]:

$$\begin{aligned} x_{(1)k+1} &= x_{(2)k}, \\ x_{(2)k+1} &= x_{(1)k} + 3x_{(1)k}^3 + x_{(2)k} + \text{sat}(u_k), \end{aligned} \tag{17}$$

which can be recast as DAR, as in (3), with $\pi_k = x^2_{(1)k}$,

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 \\ 3x_{(1)k} \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\Omega_1 = \begin{bmatrix} x_{(1)k} & 0 \end{bmatrix}, \; \Omega_2 = -1, \; \Omega_3 = 0.$$

For $u_0 = 1$, as in [13], the optimization problem (16) was solved to obtain the largest admissible polytope in state space and the largest estimated DoA. These results are compared with those from [13] and [14] in Table I, which presents the largest estimated DoA obtained from each methodology.

TABLE I: Estimated DoA for system (17) with $u_0 = 1$.

Method	Polytopic Region ($\mathcal{X}$)	$\log(\det(Q))$
Theorem 1 in [13]	$\lvert x_{(1)k}\rvert \leq 0.50$, $\lvert x_{(2)k}\rvert \leq 0.40$	-3.6952
Corollary 2 in [14]	$\lvert x_{(1)k}\rvert \leq 0.66$, $\lvert x_{(2)k}\rvert \leq 0.65$	-1.7474
Corollary 2	$\lvert x_{(1)k}\rvert \leq 0.68$, $\lvert x_{(2)k}\rvert \leq 0.67$	$\mathbf{-1.6616}$

From Table I, it is possible to verify that the proposed approach provides a larger estimated ellipsoidal DoA, in comparison with the results for DAR models presented previously in the literature.

Figure 1 depicts the estimated DoA obtained from Corollary 2, with some trajectories initiating inside this region. Notice that all trajectories starting at the boundary of the DoA converge to the origin.

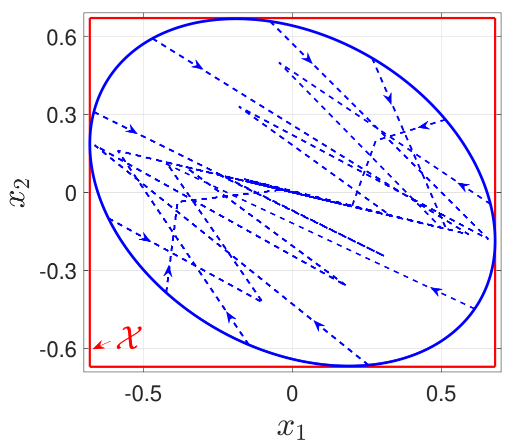

Fig. 1: Estimated DoA (blue solid line) and some state trajectories (blue dashed line) for system (17), using Corollary 2.

Example 2: Consider the discrete-time nonlinear system with disturbance input as presented in [14]:

$$\begin{aligned} x_{(1)k+1} &= x_{(2)k} + 0.22w_k, \\ x_{(2)k+1} &= (1 + x^2_{(1)k})x_{(1)k} + x_{(2)k} + 0.3w_k + \text{sat}(u_k), \\ z_k &= x_{(1)k} + x_{(2)k}. \end{aligned}$$

with a DAR given as in (3) such that

$$\pi_k = x^2_{(1)k}, A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 \\ x_{(1)k} \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} 0.22 \\ 0.3 \end{bmatrix}, \Omega_1 = \begin{bmatrix} x_{(1)k} \\ 0 \end{bmatrix}^\top, \Omega_2 = -1, \; \Omega_3 = \Omega_4 = 0,$$
$$C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; C_2 = C_3 = C_4 = 0.$$

Defining $\mathcal{X} := \{x_k \in \mathbb{R}^2 : |x_k| \leq 1.0\}$ and $u_0 = 5$, the optimization problem (15) was solved for different values of λ^{-1} (admissible energy-bound for the disturbance input). Table II presents a comparison between the results obtained by the methodology proposed in this investigation and that obtained using the approach in [14].

TABLE II: Disturbance attenuation (γ) for $u_0 = 5$.

λ^{-1}	1.5	3.0	4.5	6.0	7.0
Corollary 1 in [14]	1.07	1.18	4.72	–	–
Corollary 1	**0.56**	**0.62**	**0.75**	**1.07**	**2.50**

Notice that our approach provided less conservative results. When considering the same values for λ^{-1}, it was possible to obtain lower values for γ in comparison with the results obtained in [14]. Besides, it was possible to obtain feasible results for larger values of λ^{-1}.

Example 3: In this example, the goal is to use Corollary 1, for a rational nonlinear system with time-varying parameters. Therefore, consider the following system

$$\begin{aligned} x_{(1)k+1} &= (1 - \delta_k)x_{(2)k} + 0.5f_{n_1} + 0.5w_k, \\ x_{(2)k+1} &= x_{(2)k} - x_{(1)k} + f_{n_2} + (1 - \delta_k)\text{sat}(u_k) + 0.1w_k, \\ z_k &= x_{(1)k} + x_{(2)k}, \end{aligned} \tag{18}$$

with the nonlinear functions $f_{n_1} = x^2_{(1)k}/1 + x^2_{(1)k}$ and $f_{n_2} = x_{(1)k}/1 + x^2_{(1)k}$.

This system can be recast in a DAR form (3) such that

$$\pi_k = \begin{bmatrix} f_{n_1} & f_{n_2} \end{bmatrix}^T, \; A_1 = \begin{bmatrix} 0 & 1 - \delta_k \\ -1 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 \\ 1 - \delta_k \end{bmatrix}, \; A_4 = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix},$$
$$\Omega_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \; \Omega_2 = \begin{bmatrix} 1 & -x_{(1)k} \\ x_{(1)k} & 1 \end{bmatrix},$$
$$\Omega_3 = \Omega_4 = 0_{2\times 1}, \; C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \; C_2 = C_3 = C_4 = 0.$$

For $\mathcal{X} := \{x_k \in \mathbb{R}^2 : |x_{(1)k}| \leq 2.0, |x_{(2)k}| \leq 1.0\}$, $\Delta := \{\delta_k \in \mathbb{R} : |\delta_k| \leq 0.5\}$, and $u_0 = 2$ the optimization problem (15) was solved. Figure 2 depicts the level sets $\mathcal{E}(P_l^{-1}, \lambda^{-1})$, $l = 1, 2$, considering $\lambda^{-1} = 8.48$, with two trajectories, for $x_0 = 0$, different time-varying sequences for $\delta_k \in \Delta$ (chosen randomly), and different input disturbances $w_k^{(1)}$ and $w_k^{(2)}$ as follows

$$w_k^{(1)} = \begin{cases} -e^{(0.22k)}, & 1 \leq k \leq 3 \\ 0, & \text{elsewhere} \end{cases},$$
$$w_k^{(2)} = \begin{cases} \sqrt{8.48}, & k = 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Notice that the state trajectories do not leave the non-ellipsoidal region ($\mathcal{E}_V$), and when the disturbance vanishes, the system states converge to the origin, even when the largest admissible disturbance amplitude is applied in a single time instant (in the case of $w_k^{(2)}$).

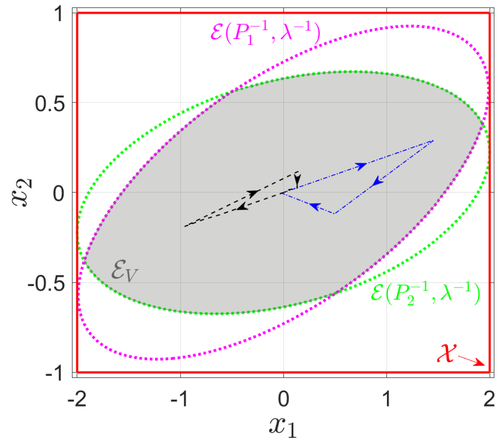

Fig. 2: Estimated non-ellipsoidal region ($\mathcal{E}_V$) for system (18), ellipsoidal regions $\mathcal{E}(P_1^{-1}, \lambda^{-1})$ (magenta dotted line) and $\mathcal{E}(P_2^{-1}, \lambda^{-1})$ (green dotted line) obtained from Corollary 1 for $\lambda^{-1} = 8.48$, and two trajectories for $x_0 = 0$, $w_k^{(1)}$ (black dashed line) and $w_k^{(2)}$ (blue dashed dotted line).

V. CONCLUSIONS

This paper has proposed new conditions based on LMIs to compute gain-scheduled controllers with a Domain of Attraction (DoA) estimation for discrete-time nonlinear systems. The class of rational systems subject to input saturation, described in a Difference-Algebraic Representation (DAR) form, is considered. In this approach, the system's nonlinearities are taken into account in the control law, showing favorable results compared to other approaches in the literature. Three numerical examples illustrated the effectiveness and advantages of the proposed method.

APPENDIX

A. Useful Lemmas

Lemma 1 (Polytopic description of input saturation [2]): Assume that the set $\mathcal{D} := \{D_r \in \mathbb{R}^{n_u \times n_u} : r \in \mathcal{I}_{N_u}\}$ is a set of diagonal matrices D_r whose diagonal elements are either 0 or 1, such that $N_u = 2^{n_u}$. Denoting $D_r^- = I_{n_u} - D_r$, one can see that $D_r^- \in \mathcal{D}$. Therefore, given any vector $v_k \in \mathbb{R}^{n_u}$, whose components satisfy $|v_{(s)k}| \leq u_{0(s)}, \forall s \in \mathcal{I}_{n_u}$, it is always possible to write

$$\text{sat}(u_k) \in \text{co}\left\{D_r u_k + {D_r}^- v_k : r \in \mathcal{I}_{N_u}\right\}. \tag{19}$$

Lemma 2 (Klug et. al [15]): The unforced system (1), with $x_0 = 0$, is input-to-output locally stable and there exists an upper bound γ on the ℓ_2-gain from w_k to z_k if (20) holds $\forall x_k \in \mathcal{L}_\mathcal{V}, \forall \delta_k \in \Delta$ and $w_k \in \mathcal{W}$:

$$\Delta V_k + \frac{1}{\gamma^2} z_k^\top z_k - w_k^\top w_k < 0. \tag{20}$$

Lemma 3 (LMI relaxations adapted from [14]): Suppose Ψ_{ijlm}^{nr}, with $i, j \in \mathcal{I}_{N_x}, l, m, n \in \mathcal{I}_{N_\delta}$, and $r \in \mathcal{I}_{N_u}$, are matrices of appropriate dimensions, such that

$$\Psi(x_k, \delta_k, \delta_{k+1}) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_x} \sum_{l=1}^{N_\delta} \sum_{m=1}^{N_\delta} \sum_{n=1}^{N_\delta} \sum_{r=1}^{N_u} \alpha_k \Psi_{ijlm}^{nr} < 0. \tag{21}$$

with $\alpha_k = \alpha_{x_{(i)k}} \alpha_{x_{(j)k}} \alpha_{\delta_{(l)k}} \alpha_{\delta_{(m)k}} \alpha_{\delta_{(n)k+1}} \alpha_{d_{(r)k}}$.

If the following LMIs hold for all $i, j \in \mathcal{I}_{N_x}$, $l, m, n \in \mathcal{I}_{N_\delta}$ and $r \in \mathcal{I}_{N_u}$

$$\Psi_{iill}^{nr} < 0,\ i = j,\ l = m,$$
$$\Psi_{ijll}^{nr} + \Psi_{jill}^{nr} < 0,\ i < j,\ l = m,$$
$$\Psi_{iilm}^{nr} + \Psi_{iiml}^{nr} < 0,\ i = j,\ l < m,$$
$$\Psi_{ijlm}^{nr} + \Psi_{ijml}^{nr} + \Psi_{jilm}^{nr} + \Psi_{jiml}^{nr} < 0,\ i < j,\ l < m,$$

then inequality (21) is satisfied.

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