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Stabilization of rational nonlinear discrete-time systems by state feedback and static output feedback

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ABSTRACT

The design of State Feedback (SF) and Static Output Feedback (SOF) controllers for nonlinear discrete-time systems subject to time-varying parameters is discussed in the context of Difference-Algebraic Representations (DAR) and parameter-dependent Lyapunov functions applied to obtain convex conditions in the form of Linear Matrix Inequalities (LMI). The proposed conditions guarantee the system robust stabilization and provide an estimate of the Domain-of-Attraction (DoA). Firstly, a novel strategy for gain-scheduled SF control is proposed incorporating information on the system's nonlinearities to compute the control action. Secondly, a new gain-scheduled SOF control design solution is derived, without structural constraints imposed on the output matrix and without making use of iterative algorithms, unlike most approaches in the current literature. Finally, numerical examples illustrate the proposed methodology's potential.

1. Introduction

Most dynamical systems in real-world applications present nonlinear behavior, such as electromechanical [42,47], electronic [37], chemical [16], and biological [6] systems. Generally, the stability analysis and the control design for nonlinear systems are very challenging. However, considering the nonlinear characteristics of certain systems is essential to model nonlinear phenomena and ensure the validity of results beyond the vicinity of equilibrium points [18]. In addition, the use of nonlinear control strategies can be important to achieve better performance for the closed-loop control system than what can be achieved by using linear techniques [41].

As a consequence of the benefits of addressing nonlinear aspects in control systems, the development of analysis and synthesis conditions for nonlinear systems has received a lot of attention in the last decades. The majority of recently proposed approaches use Linear Matrix Inequality (LMI)-based tools [39] for stability analysis and control design based on Lyapunov Stability Theory [18]. In this context, an approach widely explored by researchers is to consider a compact region in the state space in which an estimated Domain of Attraction (DoA) can be determined, in which the asymptotic stability of the system is guaranteed [1,2,4,10,23,26,33,34,43]. Since constraints on system states usually have to be enforced in practical applications due to physical limitations, these approaches are quite promising. In this investigation, we are particularly interested in the regional stabilization of a class of discrete-time nonlinear systems subject to time-varying parameters.

The class of systems considered in this research covers all systems that can be modeled as a Difference-Algebraic Representation (DAR) [9], also called Recursive-Algebraic Representation (RAR) [4,23] – the discrete-time counterpart of the Differential-Algebraic Representations [8,43,44]. This representation allows us to systematically account for rational nonlinearities. The motivation to investigate rational systems is their use to model a wide range of physical phenomena by relying on first principles or through the application of nonlinear system identification and realization theory [40]. Nevertheless, the use of DARs was not extensively explored in control theory for discrete-time systems. Contributions in this field include stability analysis [4,24], State Feedback (SF) control design [5,35], and filter design [9].

In the context of stability analysis, less conservative results were obtained by searching for polynomial and rational Lyapunov candidate functions [4]. However, due to inherent difficulties in the

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development of synthesis conditions, approaches to DAR initially used quadratic Lyapunov functions, and linear SF controllers to robust stabilization and DoA estimation [23]. Recently improvements were reported in Reis et al. [35] based on the use of parameter-dependent Lyapunov functions and gain-scheduled SF controllers. One of the purposes of this paper is to present further developments upon these previous works, reducing the conservativeness by incorporating information about the system's nonlinearities in the control law.

Another purpose of this investigation is to address the design of Static Output Feedback (SOF) controllers, a challenging stabilization problem not well addressed in the context of DARs. The SOF control design problem has received a lot of attention in the past years because it is simple to be implemented in practical situations where only partial state information is available in real-time [21,30]. However, the design of SOF control schemes is considered to be harder to solve due to its nonconvex characterization, even in the context of linear systems [31,32]. Most results are restrictive and conservative. For instance, some methodologies require a constant output matrix or particular similarity transformations [12,14]. Besides that, it is possible to find in the literature methodologies based on iterative algorithms or stabilization conditions that rely on the scalar search of different parameters, which increases the computational effort [20,36,46]. There are also two-step approaches, where the first step consists of searching for an SF controller and then the SOF control is obtained from the initial results [8,13]. More recently, in Peixoto et al. [27,28] it was proposed an alternative one-step approach to compute scheduled output-feedback control gains for discrete-time nonlinear parameter-varying systems with time-varying delay in the state and also the case for fuzzy systems [29].

Based on the previous discussions, this paper proposes novel stabilization conditions to design gain-scheduled SF and SOF controllers for rational nonlinear discrete-time systems with time-varying parameters described in a DAR form. The proposed conditions are presented in the form of LMIs obtained by considering parameter-dependent Lyapunov functions and they provide an estimate of the closed-loop DoA. Our methodology consists in one-step approach such that no iterative algorithms are required, and auxiliary decision variables are introduced only aiming at less conservative results. More specifically, our main contributions can be summarized as follows:

- A novel sufficient condition to design nonlinear gain-scheduled SF controllers for regional stabilization of discrete-time nonlinear systems is provided. Compared to other proposed methodologies for DARs in a similar context [23,35], the novelty of this approach is the use of information on the system's nonlinearities to synthesize the control law. The proposed technique is relatively simple for control design and implementation and can drastically reduce the conservativeness of the results, as illustrated by numerical examples.
- A new sufficient condition for regional stabilization of discrete-time nonlinear systems by gain-scheduled SOF controllers, albeit not explored in the context of DARs for discrete-time nonlinear systems, is presented. In this case, previous works in the context of Linear Parameter Varying (LPV) Systems have inspired our research, as for instance [30,31], in which no congruence transformations are necessary. However, we present a new methodology in which the control approach can be applied to nonlinear systems with parameter-dependent and/or nonlinear output matrix.

Hereafter, the paper is organized as follows. Section 2 presents the problem formulation. The conditions to synthesize nonlinear gain-scheduled SF controllers are given in Section 3. In Section 4, the gain-scheduled SOF control design methodology is presented. Section 5 brings important aspects to the implementation of the proposed control laws in specific situations. Numerical examples are provided in Section 6. Finally, concluding remarks are given in Section 7.

Notation: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{m\times n}$ is the set of $m\times n$ real matrices, I_n is the $n\times n$ identity matrix and diag$\{\ldots\}$ stands for a block-diagonal matrix. For a real matrix M, M^T denotes its transpose and $M>0$ $(M\geq 0)$ means that M is symmetric positive definite (semi-positive definite) and $M_{(i)}$ is the ith row. For a symmetric block matrix, the symbol $\star$ stands for the transpose of the blocks outside the main diagonal block. Let $\mathcal{I}_n=[1,\ n]\ \subset\ \mathbb{N},\ n\ \in\ \mathbb{N}$. For two sets $\mathcal{X}\subset\mathbb{R}^{n_x}$ and $\Delta\subset\mathbb{R}^{n_\delta}$, the notation $\mathcal{X}\times\Delta\subset\mathbb{R}^{n_x+n_\delta}$ is the cartesian product of $\mathcal{X}$ and Δ. $\Lambda_1\ :=\ \left\{\alpha_{\mathrm{p}_k}\in\mathbb{R}^N:\sum_{\mathrm{v}=1}^N\alpha_{\mathrm{p}_{(\mathrm{v})k}}=1,\ \alpha_{\mathrm{p}_{(\mathrm{v})k}}\geq 0\right\}$ represents the unitary simplex, where p represents an index used to distinguish different polytopes, N is the number of vertices and $\alpha_{\mathrm{p}_{(\mathrm{v})k}}$ is the vth entry in the vector at time k. Finally, the following notation is adopted to represent matrices of affine functions of (x_k,δ_k):

$$M(x_k,\delta_k)=\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta}\alpha_{x_{(i)k}}\alpha_{\delta_{(l)k}}M_{il}.$$

where N_x and N_δ denote the number of vertices of the x_k and δ_k polytopes, respectively.

2. Problem statement

Consider the following class of discrete-time nonlinear systems:

$$\begin{aligned} x_{k+1} &= f(x_k,\delta_k)+g(x_k,\delta_k)u_k,\\ y_k &= h(x_k,\delta_k)=C(x_k,\delta_k)x_k, \end{aligned}\tag{1}$$

where $x_k\in\mathcal{X}\subseteq\mathbb{R}^{n_x}$ is the state vector of the system, $\delta_k\in\Delta\subseteq\mathbb{R}^{n_\delta}$ is a time-varying parameter vector, which is available online to the controller, $u_k\ \in\ \mathbb{R}^{n_u}$ is the control input, $y_k\in\mathbb{R}^{n_y}$ is the measurement output, and $C(x_k,\delta_k)\in\mathbb{R}^{n_y\times n_x}$ is the output matrix.

In this research, we assume that functions $f(\cdot):\mathbb{R}^{n_x}\times\mathbb{R}^{n_\delta}\to\mathbb{R}^{n_x}$ (with $f(0,\delta_k)=0$) $g(\cdot):\mathbb{R}^{n_x}\times\mathbb{R}^{n_\delta}\to\mathbb{R}^{n_x\times n_u}$, and $h(\cdot):\mathbb{R}^{n_x}\times\mathbb{R}^{n_\delta}\to\mathbb{R}^{n_y}$ are rational functions well-posed on $\mathcal{X}\times\Delta$. This assumption regards the class of *rational systems* and guarantees the existence and uniqueness of the solutions of the difference equation in a neighborhood $\mathcal{X}\times\Delta$ of the equilibrium point $f(0,\delta_k)=0,\ \forall\delta_k\in\Delta$.

It is well known that a Difference-Algebraic Representation (DAR) can represent the class of rational systems in the discrete-time domain [7,8,24,25,44]. Thus, system (1) can be recast as a DAR given by

$$\begin{aligned} x_{k+1} &= A_1(x_k,\delta_k)x_k+A_2(x_k,\delta_k)\pi_k+A_3(x_k,\delta_k)u_k,\\ 0 &= \Omega_1(x_k,\delta_k)x_k+\Omega_2(x_k,\delta_k)\pi_k+\Omega_3(x_k,\delta_k)u_k,\\ y_k &= C_1(x_k,\delta_k)x_k+C_2(x_k,\delta_k)\pi_k, \end{aligned}\tag{2}$$

where $\pi_k:=\pi(x_k,\delta_k,u_k)\in\mathbb{R}^{n_\pi}$ is an auxiliary vector of nonlinear functions with respect to (x_k,δ_k) and affine with respect to (u_k). The matrices $A_1(x_k,\delta_k)\in\mathbb{R}^{n_x\times n_x}$, $A_2(x_k,\delta_k)\in\mathbb{R}^{n_x\times n_\pi}$, $A_3(x_k,\delta_k)\in\mathbb{R}^{n_x\times n_u}$, $\Omega_1(x_k,\delta_k)\in\mathbb{R}^{n_\pi\times n_x}$, $\Omega_2(x_k,\delta_k)\in\mathbb{R}^{n_\pi\times n_\pi}$, $\Omega_3(x_k,\delta_k)\in\mathbb{R}^{n_\pi\times n_u}$, $C_1(x_k,\delta_k)\in\mathbb{R}^{n_y\times n_x}$, and $C_2(x_k,\delta_k)\in\mathbb{R}^{n_y\times n_\pi}$ are affine functions of (x_k,δ_k), such that $\Omega_2(x_k,\delta_k)$ is a square full-rank matrix for all $(x_k,\delta_k)\in\mathcal{X}\times\Delta$.

The correctness of the DAR could be verified by replacing the nonlinearity vector π_k given by the null algebraic equation in (2) with the corresponding expression bellow so that (1) is obtained.

$$\pi_k=-\Omega_2^{-1}(x_k,\delta_k)[\Omega_1(x_k,\delta_k)x_k+\Omega_3(x_k,\delta_k)u_k].\tag{3}$$

It is important to point out that the decomposition of the nonlinear system in a DAR form is not unique, which can lead to conservative results. In this paper, to reduce this potential conservativeness, we use the concept of *linear annihilator*, represented in this paper as $\aleph_x(x_k) \in \mathbb{R}^{n_q \times n_x}$ (see Appendix A), and originally proposed in Trofino and Dezuo [45] for DAR models.

More often than not, e.g. due to physical limitations or associated with a validity region for the system mathematical model, one must take into consideration a domain of operation for the system states. In this context, in this investigation the state trajectories of system (2) will be considered to evolve in the following polyhedral set (which will be turned into a positively invariant set by control design):

$$\mathcal{X} := \left\{x_k \in \mathbb{R}^{n_x} : a_p^T x_k \leq 1, \quad p \in \mathcal{I}_{n_e}\right\}, \tag{4}$$

where $a_p \in \mathbb{R}^{n_x}$ is a constant n_x-dimensional vector of parameters, and n_e is the number of hyperplanes that characterize the region $\mathcal{X}$.

This research is concerned with developing LMI-based conditions that provide the stabilization of system (1) using the representation in (2). To achieve the main purpose of this investigation, the following Lyapunov function candidate is considered:

$$V(x_k, \delta_k) = x_k^T P(\delta_k) x_k, \quad P(\delta_k) = \sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} P_l, \; P_l = P_l^T > 0. \tag{5}$$

The level set associated with the function (5) is defined by

$$\mathcal{L}_{DoA} := \{x_k \in \mathbb{R}^{n_x} : V(x_k, \delta_k) \leq 1, \quad \forall \delta_k \in \Delta\}. \tag{6}$$

Lemma 1 (adapted from Jungers and Castelan [17])**.** *The level set (6) associated with the function (5) verifies that*

$$\mathcal{L}_{DoA} = \bigcap_{l \in \{1,\ldots,N_\delta\}} \mathcal{E}(P_l, 1), \tag{7}$$

with $\mathcal{E}(P_l, 1) := \left\{x_k \in \mathbb{R}^{n_x} : x_k^T P_l x_k \leq 1\right\}$.

Proof. $x_k \in \mathcal{L}_{DoA} \Leftrightarrow \forall \delta_k \in \Delta, V(x_k, \delta_k) < 1 \Leftrightarrow x_k \in \bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1)$. Moreover,

$$\bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1) \subset \bigcap_{l \in \{1,\ldots,N_\delta\}} \mathcal{E}(P_l, 1)$$

To prove that

$$\bigcap_{l \in \{1,\ldots,N_\delta\}} \mathcal{E}(P_l, 1) \subset \bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1),$$

consider $x_k \in \bigcap_{l \in \{1,\ldots,N_\delta\}} \mathcal{E}(P_l, 1)$, then $x_k^T P_l x_k < 1, \; l \in \mathcal{I}_{N_\delta}$.

Since $\sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} = 1$, the above inequality can be recast as

$$x_k^T \left(\sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} P_l\right) x_k < 1, \quad l \in \mathcal{I}_{N_\delta}.$$

Thus,

$$x_k^T P(\delta_k) x_k < 1, \quad \delta_k \in \Delta.$$

This implies that $x_k \in \mathcal{E}(P(\delta_k), 1)$, or $x_k \in \bigcap_{\delta_k \in \Delta} \mathcal{E}(P(\delta_k), 1)$. □

If $\Delta V = V(x_{k+1}, \delta_{k+1}) - V(x_k, \delta_k) < 0, \quad \forall x_k \in \mathcal{L}_{DoA}$, then (5) is said to be a Lyapunov function and $\mathcal{L}_{DoA}$ is a contractive invariant set with respect to the closed-loop system, which ensures that for $x_0 \in \mathcal{L}_{DoA}$, $x_k \to 0$, when $k \to \infty$.

By considering system (1) in a DAR form (2), this work is particularly concerned with proposing sufficient conditions to design state and static output-feedback controllers such that $\mathcal{L}_{DoA} \subset \mathcal{X}, \; \forall \delta_k \in \Delta$, is an invariant set with respect to the closed-loop, and $\mathcal{L}_{DoA}$ is as large as possible.

3. State feedback control

This section presents novel stabilization conditions for discrete-time nonlinear systems, considering all aspects stated previously. The novelty of the proposed conditions is related to the fact that the information about the nonlinearity vector π_k is incorporated in the control law. Thus, we have

$$u_k = G^{-1}(x_k, \delta_k) K(x_k, \delta_k) \xi_k, \tag{8}$$

with $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times (n_x + n_\pi)}$ and $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$ matrices of affine functions with respect to (x_k, δ_k), to be determined, and $\xi_k = \begin{bmatrix} x_k^T & \pi_k^T \end{bmatrix}^T$.

Remark 1. Notice that, we consider both the system model and the control input represented from the same basis function π_k. However, the elements of π_k that do not appear in the system representation can be removed by nulling the respective columns of the DAR matrix $A_2(x_k, \delta_k)$. On the other hand, it is possible to remove the elements of π_k that we do not want at the control input by nulling the respective columns of matrix $K(x_k, \delta_k)$ in the control input (8). For instance, the proposed control law includes the particular case

$$u_k = G^{-1}(x_k, \delta_k) \bar{K}(x_k, \delta_k) x_k, \tag{9}$$

by considering $K(x_k, \delta_k) = \begin{bmatrix} \bar{K}(x_k, \delta_k) & 0 \end{bmatrix}$ in (8), with $\bar{K}(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_x}$.

In the sequel, sufficient conditions to compute the SF control matrices that stabilize the nonlinear system (1) are presented.

Theorem 1. *Consider the nonlinear system (1) and its DAR (2). Let ϵ be a given positive scalar. If there exist matrices $P(\delta_k) = P^T(\delta_k) > 0$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, $L(x_k, \delta_k) \in \mathbb{R}^{(2n_x + n_u + n_\pi) \times (n_x + n_\pi + n_q)}$, $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$, and $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times (n_x + n_\pi)}$, such that the following inequalities hold*

$$\Gamma_1(x_k, \delta_k, \delta_{k+1}) + L(x_k, \delta_k)\Gamma_2(x_k, \delta_k) + \Gamma_2^T(x_k, \delta_k) L^T(x_k, \delta_k) < 0, \tag{10}$$

$$\begin{bmatrix} 1 & \star \\ a_p & P(\delta_k) \end{bmatrix} \geq 0, \quad p \in \mathcal{I}_{n_e}, \tag{11}$$

and

$$G^T(x_k, \delta_k) + G(x_k, \delta_k) > 0, \tag{12}$$

where

$$\Gamma_1 = \begin{bmatrix} P(\delta_{k+1}) & \star & \star \\ \epsilon K^T(x_k, \delta_k) A_3^T(x_k, \delta_k) & -P_a(\delta_k) & \star \\ -\epsilon G^T(x_k, \delta_k) A_3^T(x_k, \delta_k) & 0 & 0 \end{bmatrix},$$

$$P_a(\delta_k) = \begin{bmatrix} P(\delta_k) & \star \\ 0 & 0 \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} -I & A_1(x_k, \delta_k) & A_2(x_k, \delta_k) & A_3(x_k, \delta_k) \\ 0 & \Omega_1(x_k, \delta_k) & \Omega_2(x_k, \delta_k) & \Omega_3(x_k, \delta_k) \\ 0 & \aleph_x(x_k) & 0 & 0 \end{bmatrix},$$

then there exist a Lyapunov function (5) and a controller (8) such that, $\forall x_0$ inside $\mathcal{L}_{DoA}$ and $\delta_k \in \Delta$, the trajectory of x_k converge to the origin when $k \to \infty$.

Proof. Inequality (10) can be recast as

$$\underbrace{\Xi_1 + J\Xi_2 + \Xi_2^T J^T}_{\Gamma_1} + L\Gamma_2 + \Gamma_2^T L^T < 0, \tag{13}$$

with $\Xi_2 = \begin{bmatrix} 0 & G^{-1}(x_k, \delta_k) K(x_k, \delta_k) & -I \end{bmatrix}$,

$$\Xi_1 = \begin{bmatrix} P(\delta_{k+1}) & \star & \star \\ 0 & -P_a(\delta_k) & \star \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } J = \begin{bmatrix} \epsilon A_3(x_k, \delta_k) G(x_k, \delta_k) \\ 0 \\ 0 \end{bmatrix}.$$

Defining $\zeta = \begin{bmatrix} x_{k+1}^T & \xi_k^T & u_k^T \end{bmatrix}^T$, one has that $\Xi_2\zeta = 0$ and $\Gamma_2\zeta = 0$. By pre- and post-multiplying (13) by ζ^T and its transpose, respectively, results in $\Delta V_k = x_{k+1}^T P(\delta_{k+1})x_{k+1} - x_k^T P(\delta_k)x_k < 0$. This proves that if the condition (10) is feasible, then $V(x_k, \delta_k)$ is a Lyapunov function and the controller (8) ensures that the origin of the closed-loop system is asymptotically stable.

Multiplying (11) with $\begin{bmatrix} 1 & -x_k^T \end{bmatrix}$ on the left and its transpose on the right, yields

$$1 - x_k^T a_p - a_p^T x_k + x_k^T P(\delta_k)x_k \geq 0.$$

Since $x_k^T P(\delta_k)x_k \leq 1$ for all $x \in \mathcal{L}_{DoA}$, this inequality implies that $a_p^T x_k \leq 1$. This guarantees the inclusion $\mathcal{L}_{DoA} \subseteq \mathcal{X}$.

Finally, condition (12) ensures the existence of the inverse of matrix $G(x_k, \delta_k)$, $\forall x_k \in \mathcal{X}$ and $\delta_k \in \Delta$, which is necessary to guarantee the computation of the control law in (8). □

An alternative to find the largest DoA is to consider the following subset of $\mathcal{L}_{DoA}$

$$\mathcal{E}(Q, 1) \subseteq \bigcap_{l \in \{1,\ldots,N_\delta\}} \mathcal{E}(P_l, 1). \tag{14}$$

In the next Corollary, Theorem 1 can be used to maximize the estimated DoA from a Semidefinite Programming (SDP) problem subjected to LMI conditions.

Corollary 1. *Given a positive scalar $\epsilon > 0$. If there exist symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $P(\delta_k) > 0$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, and any matrices $L(x_k, \delta_k) \in \mathbb{R}^{(2n_x+n_u+n_\pi)\times(n_x+n_\pi+n_q)}$, $G(x_k, \delta_k) \in \mathbb{R}^{n_u \times n_u}$, and $K(x_k, \delta_k) \in \mathbb{R}^{n_u \times (n_x+n_\pi)}$, satisfying the following optimization problem for all $\delta_k \in \Delta$ and $x_k \in \mathcal{X}$:*

$$\begin{cases} \min \{\mathrm{trace}(Q)\} \\ \text{subject to (10)–(12), and } Q - P(\delta_k) > 0, \end{cases} \tag{15}$$

then the SF controller (8) asymptotically stabilizes the closed-loop system, composed to (1) and (8), around the origin, and $\mathcal{E}(Q, 1) \subseteq \mathcal{L}_{DoA}$ is an estimate of the DoA.

Proof. The additional inequality in (15) ensures that $\mathcal{E}(Q, 1) \subseteq \mathcal{L}_{DoA}$, which is defined in (7), and the rest of the proof follows in a straightforward way as in the proof of Theorem 1. □

4. Static output feedback control

Section 3 presented the conditions to design an SF controller for regional stabilization of discrete-time nonlinear systems when full state information is available. In this section, considering practical applications in which only the system output is measurable in real-time, our goal is to design an SOF controller in the form

$$u_k = F^{-1}(\delta_k)H(\delta_k)y_k, \tag{16}$$

with $H(\delta_k) \in \mathbb{R}^{n_u \times n_y}$ and $F(\delta_k) \in \mathbb{R}^{n_u \times n_u}$ matrices of affine functions with respect to (δ_k), to be determined.

Remark 2. Since SOF controllers are an alternative more explored in practical situations where the complete state information is not available for real-time control implementation, the information about the system states vector is not taken into account in the gain matrices $F(\cdot)$ and $H(\cdot)$, which are only dependent on parameters (δ_k).

Theorem 2 in the sequel presents a new SOF control design for the nonlinear system (1).

Theorem 2. *Consider the nonlinear system (1) and its DAR (2). Let ϵ be a given positive scalar. If there exist matrices $P(\delta_k) = P^T(\delta_k) > 0$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, $S(x_k, \delta_k) \in \mathbb{R}^{(2n_x+n_y+n_u+n_\pi)\times(n_x+n_y+n_\pi+n_q)}$, $F(\delta_k) \in \mathbb{R}^{n_u \times n_u}$, and $H(\delta_k) \in \mathbb{R}^{n_u \times n_y}$, such that the following inequalities hold*

$$\Upsilon_1(x_k, \delta_k, \delta_{k+1}) + S(x_k, \delta_k)\Upsilon_2(x_k, \delta_k) + \Upsilon_2^T(x_k, \delta_k)S^T(x_k, \delta_k) < 0, \tag{17}$$

$$\begin{bmatrix} 1 & \star \\ a_p & P(\delta_k) \end{bmatrix} \geq 0, \quad p \in \mathcal{I}_{n_e}, \tag{18}$$

and

$$F^T(\delta_k) + F(\delta_k) > 0, \tag{19}$$

where

$$\Upsilon_1 = \begin{bmatrix} -P(\delta_k) & \star & \star & \star & \star \\ 0 & P(\delta_{k+1}) & \star & \star & \star \\ \epsilon H^T(\delta_k)A_3^T(x_k, \delta_k) & H^T(\delta_k)A_3^T(x_k, \delta_k) & 0 & \star & \star \\ -\epsilon F^T(\delta_k)A_3^T(x_k, \delta_k) & -F^T(\delta_k)A_3^T(x_k, \delta_k) & \epsilon H(\delta_k) & \mathcal{A}_{44} & \star \\ 0 & 0 & \mathcal{A}_{53} & \mathcal{A}_{54} & 0 \end{bmatrix},$$

$$\mathcal{A}_{44} = -\epsilon F(\delta_k) - \epsilon F^T(\delta_k),$$
$$\mathcal{A}_{53} = \Omega_3(x_k, \delta_k)H(\delta_k),$$
$$\mathcal{A}_{54} = -\Omega_3(x_k, \delta_k)F(\delta_k),$$

$$\Upsilon_2 = \begin{bmatrix} A_1(x_k, \delta_k) & -I & 0 & A_3(x_k, \delta_k) & A_2(x_k, \delta_k) \\ C_1(x_k, \delta_k) & 0 & -I & 0 & C_2(x_k, \delta_k) \\ \Omega_1(x_k, \delta_k) & 0 & 0 & \Omega_3(x_k, \delta_k) & \Omega_2(x_k, \delta_k) \\ \aleph_x(x_k) & 0 & 0 & 0 & 0 \end{bmatrix},$$

then there exist a Lyapunov function (5) and a controller (16) such that, $\forall x_0$ inside $\mathcal{L}_{DoA}$ and $\delta_k \in \Delta$, the trajectory of x_k converge to the origin when $k \to \infty$.

Proof. Inequality (17) can be recast as

$$\Theta_1 + R\Theta_2 + \Theta_2^T R^T + S\Upsilon_2 + \Upsilon_2^T S^T < 0,$$

where $\Theta_2 = \begin{bmatrix} 0 & 0 & F^{-1}(\delta_k)H(\delta_k) & -I & 0 \end{bmatrix}$,

$$\Theta_1 = \begin{bmatrix} -P(\delta_k) & \star & \star & \star & \star \\ 0 & P(\delta_{k+1}) & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$R = \begin{bmatrix} \epsilon A_3(x_k, \delta_k)F(\delta_k) \\ A_3(x_k, \delta_k)F(\delta_k) \\ 0 \\ \epsilon F^T(\delta_k) \\ \Omega_3(x_k, \delta_k)F(\delta_k) \end{bmatrix}.$$

Defining $\vartheta = \begin{bmatrix} x_k^T & x_{k+1}^T & y_k^T & u_k^T & \pi_k^T \end{bmatrix}^T$, one has $\Theta_2\vartheta = 0$ and $\Upsilon_2\ \vartheta = 0$. Multiplying the latter inequality by ϑ^T on the left and its transpose on the right, yields $\Delta V_k = x_{k+1}^T P(\delta_{k+1})x_{k+1} - x_k^T P(\delta_k)x_k < 0$. Thus, if the condition (17) is feasible, then $V(x_k, \delta_k)$ is a Lyapunov function and the SOF controller (16) ensures that the origin of the closed-loop system is asymptotically stable.

Constraint (18) is obtained following the same steps in Theorem 1 and condition (19) ensures the existence of the inverse of matrix $F(\delta_k)$, $\forall \delta_k \in \Delta$, which is necessary to guarantee the computation of the control law in (16). □

Similarly to the previous results stated in Section 3, the next Corollary can be used in order to maximize the estimated DoA.

Corollary 2. *Consider a given positive scalar $\epsilon > 0$. If there exist symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $P(\delta_k) > 0$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, and matrices $S(x_k, \delta_k) \in \mathbb{R}^{(2n_x+n_y+n_u+n_\pi)\times(n_x+n_y+n_\pi+n_q)}$, $F(\delta_k) \in \mathbb{R}^{n_u \times n_u}$, and*

$H(\delta_k) \in \mathbb{R}^{n_u \times n_y}$ satisfying the following optimization problem for all $\delta_k \in \Delta$ and $x_k \in \mathcal{X}$:

$$\begin{cases} \min\{\text{trace}(Q)\} \\ \text{subject to (17)–(19), and } Q - P(\delta_k) > 0, \end{cases} \tag{20}$$

then the SOF controller (16) asymptotically stabilizes the closed-loop system, composed to (1) and (16), around the origin, and $\mathcal{E}(Q,1) \subseteq \mathcal{L}_{DoA}$ is an estimate of the DoA.

Proof. The additional inequality in (20) ensures that $\mathcal{E}(Q,1) \subseteq \mathcal{L}_{DoA}$ and the rest of the proof follows in a straightforward way as in the proof of Theorem 2. □

Remark 3. Notice that the proposed methodology can be applied to rational systems with nonlinear and/or parameter-dependent output matrix. Besides that, no structural constraint is imposed on the output matrix, and the SOF control design problem is solved directly, without the necessity to obtain an SF controller in the first step or use iterative algorithms, unlike other literature approaches [8,12–14,20,46]. It is worth emphasizing that the given scalar ϵ is introduced only to likely yield a less conservative result.

Remark 4. Toward developing the results from Theorems 1 and 2, we decided to adopt a parameter-dependent Lyapunov function, aiming to obtain LMI conditions which can provide less conservative results compared to the ones obtained by using standard quadratic Lyapunov functions. To potentially reduce the conservatism, enhanced Lyapunov functions are usually employed for stability analysis. In the context of discrete-time nonlinear systems described in a DAR form, to the best of the authors' knowledge, only Coutinho and de Souza [4] had proposed analysis conditions based on polynomial Lyapunov functions, but without providing synthesis conditions due to its inherent difficulties. On the other hand, in the literature, there are also methods using sum of squares (SOS) decomposition of the Lyapunov stability conditions that can be cast as a semidefinite program (SDP) [3,38], but this is not the aim in this work.

5. On the implementation of the control law

The proposed approaches so far have considered, in the system model, the presence of time-varying parameters (δ_k), which are supposed to be exactly known, and this information is used in the gain-scheduled control strategy, aiming to achieve less conservative results.

In real-world applications, a more realistic situation is the case where the dynamical system presents physical parameters that are not precisely known, i.e., the nonlinear system model includes uncertain parameters whose bounds, in many cases, are known and can be taken into account in the stabilization conditions. In this context, the proposed control laws can be adapted to deal with the parametric uncertainties associated with unknown parameters.

For SF robust control, Theorem 1 can be applied by considering the matrices $G(\cdot)$ and $K(\cdot)$ only affine with respect to states (x_k). In this case, if the nonlinearity vector, π_k, depends on part of the parametric uncertainties, we must null the respective columns of matrix $K(\cdot)$, as discussed in Remark 1. For SOF robust control, it is possible to apply Theorem 2, defining F and H as constant matrices.

Another situation that requires attention in practical applications is when $\Omega_3(x_k,\delta_k) \neq 0$, i.e., vector π_k is dependent on (u_k). Concerning the design of SF controllers, as for the case of π_k dependent on uncertain parameters, it is possible to deal with this problem by nulling the respective columns of matrix $K(\cdot)$ and solving the stabilization conditions in Theorem 1. Alternatively, the following Corollary can be used to synthesize an SF controller incorporating the complete information of vector π_k, in which the final implementation of the control law is guaranteed.

Corollary 3. *Consider a given positive scalar $\epsilon > 0$. If there exist symmetric matrices $Q \in \mathbb{R}^{n_x \times n_x}$ and $P(\delta_k) > 0$, $P(\delta_k) \in \mathbb{R}^{n_x \times n_x}$, and any matrices $L(x_k,\delta_k) \in \mathbb{R}^{(2n_x+n_u+n_\pi)\times(n_x+n_\pi+n_q)}$, $G(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_u}$, and $K(x_k,\delta_k) = \begin{bmatrix}\bar{K}(x_k,\delta_k) & \hat{K}(x_k,\delta_k)\end{bmatrix}$, $\bar{K}(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_x}$, $\hat{K}(x_k,\delta_k) \in \mathbb{R}^{n_u \times n_\pi}$ satisfying the following optimization problem for all $\delta_k \in \Delta$ and $x_k \in \mathcal{X}$:*

$$\begin{cases} \min\{\text{trace}(Q)\} \\ \text{subject to (10), (11),} \quad Q - P(\delta_k) > 0, \text{ and} \end{cases} \tag{21}$$

$$\begin{bmatrix} G^T(x_k,\delta_k) + G(x_k,\delta_k) & \hat{K}(x_k,\delta_k) + \Omega_3^T(x_k,\delta_k) \\ \hat{K}^T(x_k,\delta_k) + \Omega_3(x_k,\delta_k) & -(\Omega_2^T(x_k,\delta_k) + \Omega_2(x_k,\delta_k)) \end{bmatrix} > 0, \tag{22}$$

then the SF controller (8) asymptotically stabilizes the closed-loop system comprised by (1) and (8), around the origin, and $\mathcal{E}(Q,1) \subseteq \mathcal{L}_{DoA}$ is an estimate of the DoA.

Proof. Note that, from (8), we have

$u_k = G^{-1}(x_k,\delta_k)\big[\bar{K}(x_k,\delta_k)x_k + \hat{K}(x_k,\delta_k)\pi_k\big]$,

where $K(x_k,\delta_k) = \begin{bmatrix}\bar{K}(x_k,\delta_k) & \hat{K}(x_k,\delta_k)\end{bmatrix}$ in (8). At the same time, from (3), if $\Omega_3(x_k,\delta_k) \neq 0$ and using the assumption that $\exists \Omega_2^{-1}(x_k,\delta_k)$, one has that (dependency with (x_k,δ_k) was dropped for clarity purposes)

$u_k = G^{-1}\big[\bar{K}x_k - \hat{K}\Omega_2^{-1}(\Omega_1 x_k + \Omega_3 u_k)\big]$,

which can be rewritten as

$\big[G + \hat{K}\Omega_2^{-1}\Omega_3\big]u_k = \bar{K}x_k - \hat{K}\Omega_2^{-1}\Omega_1 x_k.$

Therefore, the final implementation of the control law (8) is given by

$$u_k = \big[G + \hat{K}\Omega_2^{-1}\Omega_3\big]^{-1}\big[\bar{K} - \hat{K}\Omega_2^{-1}\Omega_1\big]x_k, \tag{23}$$

as long as the matrix $M(x_k,\delta_k) = \big[G + \hat{K}\Omega_2^{-1}\Omega_3\big]$ is invertible.

If $\Omega_3 \equiv 0$ or $\hat{K} \equiv 0$, $M(x_k,\delta_k)$ is nonsingular from the satisfaction of (12) in Theorem 1. On the other hand, if $\Omega_3 \neq 0$ and $\hat{K} \neq 0$, from (22) one has

$$\Phi^T(x_k,\delta_k) + \Phi(x_k,\delta_k) > 0, \quad \Phi(x_k,\delta_k) = \begin{bmatrix} G & \hat{K} \\ \Omega_3 & -\Omega_2 \end{bmatrix}$$

From the feasibility of the above inequality, $\Phi(x_k,\delta_k)$ must be invertible. Notice that matrix $\Phi(x_k,\delta_k)$ can be recast as

$$\Phi(x_k,\delta_k) = \begin{bmatrix} I & -\hat{K}\Omega_2^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} G + \hat{K}\Omega_2^{-1}\Omega_3 & 0 \\ 0 & -\Omega_2 \end{bmatrix}\begin{bmatrix} I & 0 \\ -\Omega_2^{-1}\Omega_3 & I \end{bmatrix}.$$

Thus, $\det(\Phi(x_k,\delta_k)) = \det(G + \hat{K}\Omega_2^{-1}\Omega_3)\det(-\Omega_2) = \det(M)\det(-\Omega_2)$.

Therefore, if condition (22) is satisfied, then $\det(\Phi(x_k,\delta_k)) \neq 0$ and $M(x_k,\delta_k)$ is invertible. The rest of the proof follows in a straightforward way as in the proofs of Theorem 1 and Corollary 1. □

Remark 5. Notice that condition (22) in Corollary 3 requires the additional restriction that $\Omega_2^T(x_k,\delta_k) + \Omega_2(x_k,\delta_k)$ must be a negative definite matrix. In the cases where matrix $\Omega_2(\cdot)$ has a definite sign, the fact that $\Omega_2(\cdot)$ must be negative is not restrictive, as it is possible to change the sign of this matrix in the definition of the algebraic equation of the DAR, without loss of generality. If $\Omega_2(\cdot)$ has no definite sign but is constant, an alternative to guarantee

Table 1
Largest estimated DoA for system (24), from Corollary 2.

Lyapunov function	Polytopic region ($\mathcal{X}$)	Area of the estimated DoA
quadratic	$\lvert x_{(1)k}\rvert \le 0.42$, $\lvert x_{(2)k}\rvert \le 0.21$	0.2304
parameter-dependent	$\boldsymbol{\lvert x_{(1)k}\rvert \le 0.62, \lvert x_{(2)k}\rvert \le 0.32}$	**0.4836**

that the additional constraint in (22) holds is obtained multiplying the DAR algebraic equation on the left by $-\Omega_2^T(\cdot)$, as this results in a new negative definite $\Omega_2(\cdot)$ matrix. On the other hand, the cases where $\Omega_2(\cdot)$ is affine with respect to (x_k, δ_k) without a definite sign are more complex and should be analyzed carefully depending on the problem. Despite this fact, considerably less conservative results can be obtained using the complete information of the nonlinearity vector (π_k), as shown in Example 4.

6. Numerical examples

In this section, numerical examples are presented to demonstrate the effectiveness of the proposed methodology. The stabilization conditions were presented in an infinite-dimensional form, which is not computationally tractable. Thus, Appendix B gives the LMI relaxation employed to convert the conditions proposed in Theorems 1 and 2 into finite sets of LMIs, which were implemented in MATLAB (R2019) using the parser Yalmip and the solver Mosek.

Example 1. Consider the following rational nonlinear system with a time-varying parameter:

$$
\begin{aligned}
x_{(1)k+1} &= (1-\delta_{(1)k})x_{(2)k} + \frac{x_{(1)k}^2}{1+x_{(1)k}^2} - \frac{\delta_{(1)k}x_{(1)k}^4}{1+x_{(1)k}^2},\\
x_{(2)k+1} &= -x_{(1)k} + x_{(2)k} + 0.5\frac{\delta_{(1)k}x_{(1)k}}{1+x_{(1)k}^2} + (1+2\delta_{(1)k})u_k,\\
y_k &= x_{(1)k} + 2\delta_{(1)k}x_{(2)k} + \frac{x_{(1)k}^4}{1+x_{(1)k}^2}.
\end{aligned}
\tag{24}
$$

with a corresponding DAR such that

$$
\pi = \begin{bmatrix} \frac{x_{(1)k}^4}{1+x_{(1)k}^2} & \frac{x_{(1)k}^3}{1+x_{(1)k}^2} & \frac{x_{(1)k}^2}{1+x_{(1)k}^2} & \frac{x_{(1)k}}{1+x_{(1)k}^2} \end{bmatrix}^T,
$$

$$
C_1 = \begin{bmatrix} 1 & 2\delta_{(1)k} \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},
$$

$$
A_1 = \begin{bmatrix} 0 & 1-\delta_{(1)k} \\ -1 & 1 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} -\delta_{(1)k} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5\delta_{(1)k} \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1+2\delta_{(1)k} \end{bmatrix},
$$

$$
\Omega_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},\ \Omega_2 = \begin{bmatrix} -1 & x_{(1)k} & 0 & 0 \\ 0 & -1 & x_{(1)k} & 0 \\ 0 & 0 & -1 & x_{(1)k} \\ 0 & 0 & -x_{(1)k} & -1 \end{bmatrix},
$$

$$
\Omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

Defining $\Delta := \{\delta_k \in \mathbb{R} : |\delta_{(1)k}| \le 0.13\}$, the optimization problem stated in Corollary 2 was solved to design an SOF controller aiming to obtain the largest admissible polytope in the state space and the largest estimated DoA, such that system (24) can be stabilized. The results obtained from the proposed approach by considering a parameter-dependent Lyapunov function and a standard quadratic Lyapunov function are summarized in Table 1. For better clarification, the SOF controller gain matrices are described in Appendix C.

The less conservative result was obtained by defining $\epsilon = 1 \times 10^{-8}$ and $\mathcal{X} := \{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 0.62 \text{ and } |x_{(2)k}| \le 0.32\}$, to solve the optimization problem (20), considering a parameter-dependent Lyapunov function, which results in an estimated DoA with area equal to 0.4836. In this case, the minimum value for the objective function related to the area of the estimated DoA is $\text{trace}(Q) = 18.7632$.

The obtained matrices P_l are given by:

$$
P_1 = \begin{bmatrix} 3.0374 & -1.5727 \\ -1.5727 & 10.5799 \end{bmatrix},\quad P_2 = \begin{bmatrix} 3.7118 & -3.9998 \\ -3.9998 & 14.4083 \end{bmatrix}.
$$

Fig. 1 depicts the largest estimated DoA (region filled in blue) and some trajectories initiating inside it for different time-varying sequences for $\delta_k \in \Delta$. These trajectories start at the boundary of the DoA and converge to the origin.

Notice that the largest estimated DoA obtained by considering a parameter-dependent Lyapunov function is not an ellipsoid, but the intersection of the two ellipsoids (magenta and green dotted lines) associated with $\mathcal{E}(P_l, 1)$. This example highlights the non-ellipsoidal characteristic of the parameter-dependent Lyapunov function, which can provide a less conservative result in comparison with the use of standard quadratic Lyapunov functions.

Example 2. In this example, our goal is to use Corollary 1, considering a practical application with the presence of uncertain time-varying parameters. In this sense, consider the inverted pendulum model

$$
\ddot{\theta}(t) = \frac{g}{l}\sin(\theta(t)) - \frac{b\dot{\theta}(t)}{M} + \frac{\tau(t)}{Ml^2} \tag{25}
$$

where g is the gravitational acceleration, l is the length of the pendulum rod, M is the total mass and b is the damping coefficient. Besides that, $\theta(t)$ is the angle from the vertical direction and $\tau(t)$ is the control torque.

Using the change of variables $r = \arctan(\theta)$, with $\sin(\theta) = (2r)/(1+r^2)$ and $\cos(\theta) = (1-r^2)/(1+r^2)$

$$
\begin{aligned}
\dot{x}_{(1)}(t) &= x_{(2)}(t),\\
\dot{x}_{(2)}(t) &= \frac{g}{l}x_{(1)}(t) + \frac{2x_{(1)}(t)x_{(2)}^2(t)}{1+x_{(1)}^2(t)} - \frac{b}{M}x_{(2)}(t) + \frac{1+x_{(1)}^2(t)}{2Ml^2}u(t).
\end{aligned}
\tag{26}
$$

Suppose that parameters b and M have uncertainties, such that $M = M_0(1+\delta_{(1)}(t))$ and $b = b_0(1+\delta_{(2)}(t))$, with M_0 and b_0 being nominal values and the uncertain parameter vector $\delta(t) = \begin{bmatrix}\delta_{(1)}(t) & \delta_{(2)}(t)\end{bmatrix}^T$. Using Euler's first-order approximation, the following discrete-time model is obtained:

$$
\begin{aligned}
x_{(1)k+1} &= x_{(1)k} + Tx_{(2)k},\\
x_{(2)k+1} &= x_{(2)k} + T\left[\frac{g}{l}x_{(1)k} + f_n(x_k, \delta_k, u_k)\right],
\end{aligned}
\tag{27}
$$

where T is the sampling period and $f_n(\cdot)$ is a rational function with respect to (x_k, δ_k), given by

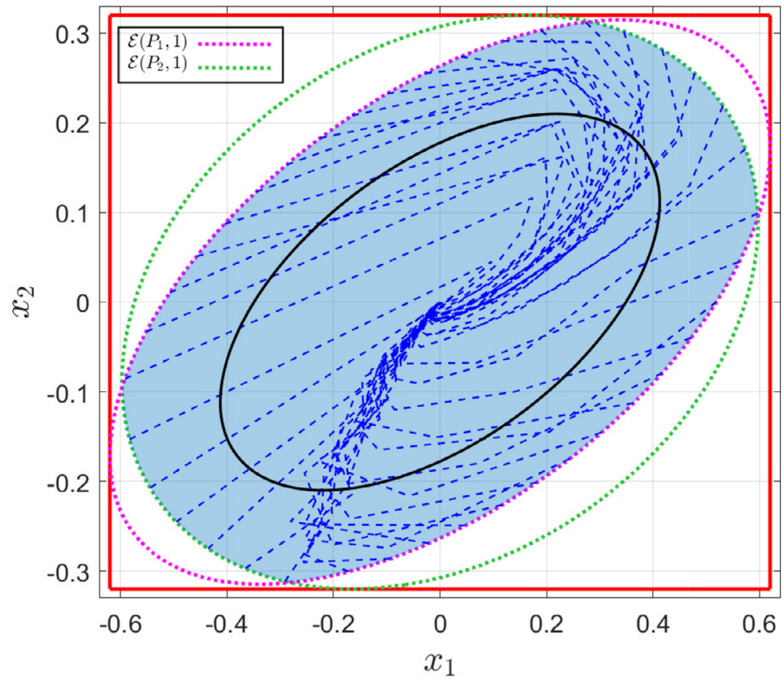

Fig. 1. Estimated DoA and some state trajectories (blue dashed lines) for system (24). $\mathcal{L}_{DoA}$ (region filled in blue) is the estimated non-ellipsoidal DoA obtained from Corollary 2, based on parameter-dependent Lyapunov function. The ellipsoidal region represented by the black solid line is the estimated DoA from Corollary 2, considering a quadratic Lyapunov function. The two ellipsoids (magenta and green dotted line) associated with $\mathcal{E}(P_i, 1)$ are also shown in this figure. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$f_n(x_k, \delta_k, u_k) = \frac{2x_{(1)k}x_{(2)k}^2}{1 + x_{(1)k}^2} - \frac{b_0(1 + \delta_{(2)k})}{M_0(1 + \delta_{(1)k})}x_{(2)k} + \frac{1 + x_{(1)k}^2}{2M_0(1 + \delta_{(1)k})l^2}u_k$$

Thus, system (27) can be recast as a DAR (2) with

$$\pi_k = \begin{bmatrix} \frac{x_{(2)k}}{1 + x_{(1)k}^2} & \frac{x_{(1)k}x_{(2)k}}{1 + x_{(1)k}^2} & \frac{x_{(2)k}}{1 + \delta_{(1)k}} & \frac{u_k}{1 + \delta_{(1)k}} & \frac{x_{(1)k}u_k}{1 + \delta_{(1)k}} \end{bmatrix}^T,$$

$$A_1 = \begin{bmatrix} 1 & T \\ \frac{Tg}{l} & 1 \end{bmatrix},$$

$$A_2 = T\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2x_{(2)k} & -\frac{b_o}{M_o}(1 + \delta_{(2)k}) & -\frac{\delta_{(1)k}}{2M_0l^2} & \frac{x_{(1)k}}{2M_0l^2} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 \\ \frac{T}{2M_0l^2} \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}^T,$$

$$\Omega_2 = \begin{bmatrix} -1 & -x_{(1)k} & 0 & 0 & 0 \\ x_{(1)k} & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 - \delta_{(1)k} & 0 & 0 \\ 0 & 0 & 0 & -1 - \delta_{(1)k} & 0 \\ 0 & 0 & 0 & x_{(1)k} & -1 \end{bmatrix}, \Omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Choosing $T = 0.05$ s and considering $M_0 = 10$ Kg, $l = 2$ m, $g = 9.8$ m/s^2, $b_0 = 0.5$ Ns/m, and $\Delta := \{\delta_k \in \mathbb{R}^2 : |\delta_{(1)k}| \le 0.1, |\delta_{(2)k}| \le 0.9\}$, the optimization problem (15) was solved for $\mathcal{X} := \{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 0.35, |x_{(2)k}| \le 0.75\}$ and $\epsilon = 1 \times 10^2$.

Since the time-varying parameters (δ_k) are not exactly known, in this case, Corollary 1 is applied by considering matrices $G(\cdot)$ and $K(\cdot)$ only affine with respect to states (x_k), as discussed in Section 5. Moreover, note that the last three elements of vector π_k are dependent on (δ_k). For this reason, the respective columns of matrix $K(\cdot)$ were nulled, such that the SF control law does not depend on (δ_k). In Appendix C, the obtained controller gain matrices are described.

Fig. 2 depicts the largest estimated DOA (region filled in blue) and some trajectories obtained by simulating the closed-loop system from Eq. (27) and the control law (8), considering different time-varying sequences for $\delta_k \in \Delta$.

In addition, zoom images at different points are presented in Fig. 2. At point 1 (top right corner), taking the DOA as a reference, there are the overlapping ellipsoids $\mathcal{E}(P_1, 1)$ (magenta), $\mathcal{E}(P_2, 1)$ (cyan), $\mathcal{E}(P_3, 1)$ (green), and $\mathcal{E}(P_4, 1)$ (orange), in this order. On the other hand, at point 2 (lower left corner), the order is reversed, which shows that there are points of intersection between these regions, as for the previous example. The estimated DoA is the intersection of these four ellipsoids.

This example shows the effectiveness of the proposed method when uncertain time-varying parameters are considered. The following numerical examples illustrate the methodology's potential, showing favorable comparisons with recently published similar approaches.

Example 3. Consider the following nonlinear system that does not have time-varying parameters, adapted from [23]:

$$\begin{aligned} x_{(1)k+1} &= x_{(2)k}, \\ x_{(2)k+1} &= x_{(1)k} + 3x_{(1)k}^3 + x_{(2)k} + u_k, \\ y_k &= x_{(1)k} + 1.2x_{(1)k}^3, \end{aligned} \tag{28}$$

which can be recast in a DAR, such that

$$\begin{aligned} &\pi_k = x_{(1)k}^2, \; A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 \\ 3x_{(1)k} \end{bmatrix}, \; A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ &\Omega_1 = \begin{bmatrix} x_{(1)k} & 0 \end{bmatrix}, \; \Omega_2 = -1, \quad \Omega_3 = 0, \\ &C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; C_2 = 1.2x_{(1)k}. \end{aligned} \tag{29}$$

Two situations are taken into account in this example. Firstly, we consider that the whole information about the system's states

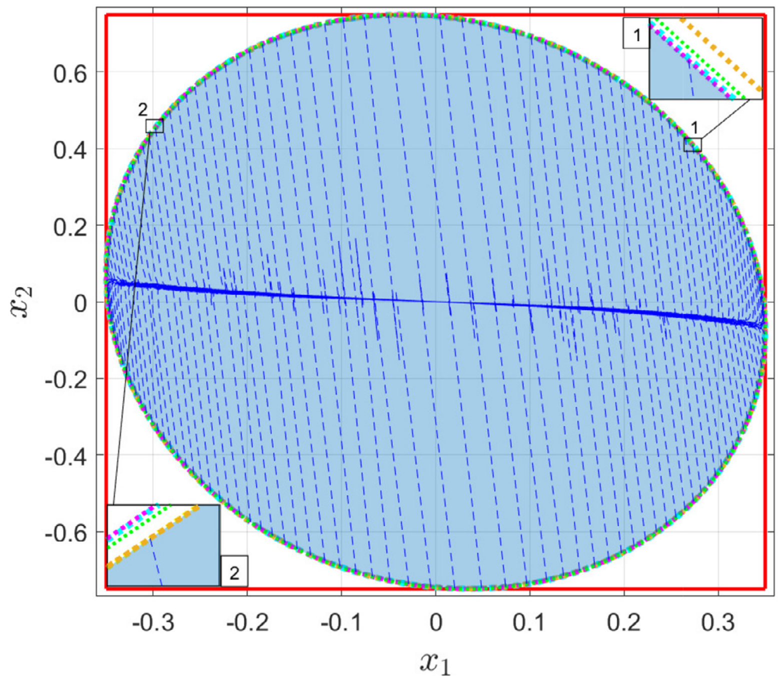

Fig. 2. Estimated DoA and some state trajectories (blue dashed lines) for system (27). $\mathcal{L}_{DoA}$ (region filled in blue) is the estimated DoA obtained from Corollary 1, based on parameter-dependent Lyapunov function. The four overlapping ellipsoids (orange, magenta, green, and cyan dotted line) associated with $\mathcal{E}(P_i, 1)$ are also shown in this figure. In addition, zoom images are presented at two points to highlight the crossing of the ellipsoids. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 2
Largest estimated DoA for system (28) obtained from Corollary 1.

Control law	Polytopic region ($\mathcal{X}$)	trace()P
Not dependent on π_k	$\lvert x_{(1)k}\rvert \le 0.81$, $\lvert x_{(2)k}\rvert \le 0.81$	3.0483
Dependent on π_k	$\mathbf{\lvert x_{(1)k}\rvert \le 10.00}$, $\mathbf{\lvert x_{(2)k}\rvert \le 10.00}$	**0.0200**

is available. In this case, Corollary 1 is used to synthesize an SF controller. Secondly, we suppose that only partial state information is measured and an SOF controller is designed by applying Corollary 2.

- Case 1: SF Control Design

The problem, in this case, is to design an SF controller in order to obtain the largest admissible polytope in state space and the largest estimated DoA, such that system (28) can be stabilized. It is worth mentioning that this example is explored in Oliveira et al. [23], Reis et al. [35] in the context of rational systems with input saturation without incorporating information about the system's nonlinearities in the control law. Although our results compared favorably with those in Oliveira et al. [23], Reis et al. [35], even when the conditions were changed to disregard saturation limits, the comparison with these works could be unfair since the control design with saturation is not taken into consideration in our current research. In this sense, the results obtained applying Corollary 1 with and without the nonlinearity vector in the control law (as discussed in Remark 1) are presented to demonstrate the potential of the proposed methodology in drastically reducing the design conservativeness.

In this analysis, we considered that the system states are limited to the polyhedral set $\overline{\mathcal{X}} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 10.0, |x_{(2)k}| \le 10.0\right\}$. Thus, the optimization problem presented in (15) was solved and the results obtained are shown in Table 2.

From Table 2, one can see that by incorporating information about the system's nonlinearities in the control law, it is possible to obtain feasible results for a considerably larger polytopic region. As a result, the largest estimated DoA is obtained from Corollary 1 with the control law dependent on π_k, which provides the smallest value for the objective function trace$(P) = .0200$, with $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 10.0, |x_{(2)k}| \le 10.0\right\}$ and $\epsilon = 1$.

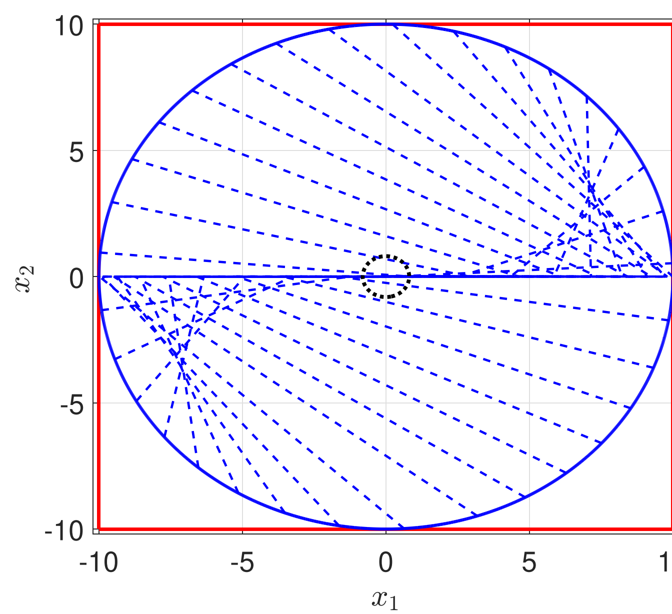

Fig. 3. Largest estimated DoA (blue solid line) and some trajectories (blue dashed line) obtained using Corollary 1 with $\mathcal{X} := \left\{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 10.0 \text{ and } |x_{(2)k}| \le 10.0\right\}$ (red solid line) and the control law dependent on π_k. The estimated DoA obtained by considering the control law, which is not dependent on π_k, is represented by the black dotted line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

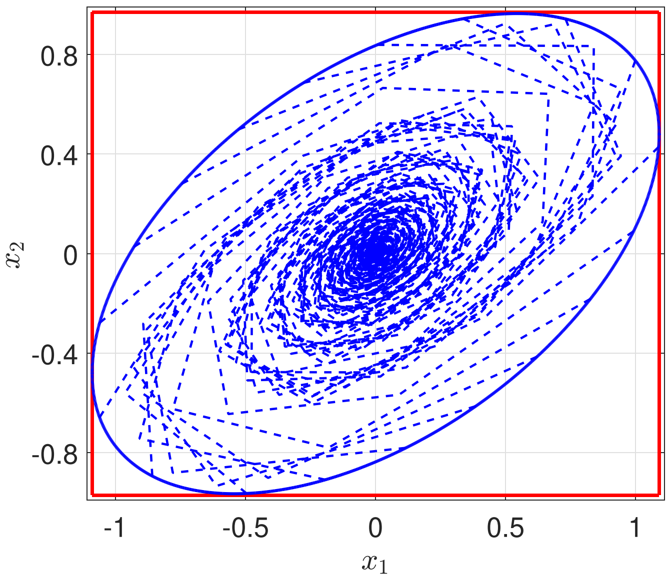

Fig. 4. Largest estimated DoA (blue solid line) and some trajectories (blue dashed line) obtained using Corollary 2 with $\mathcal{X} := \{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 1.09, |x_{(2)k}| \le 0.97\}$(red solid line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 3 depicts the largest estimated DoA (blue solid line) and some trajectories initiating inside this region. Note that all trajectories starting at the boundary of the DoA converge to the origin. Fig. 3 also shows the estimated DoA obtained from Corollary 1 with the control law not dependent on π_k (black dotted line).

Since there are no time-varying parameters, the use of a standard quadratic Lyapunov function ($P(\delta_k) = P$) might be considered. However, from Fig. 3 notice that how the estimated DoA obtained via the proposed approach is considerably less conservative when the system's nonlinearities are taken into account in the control law. The area of the largest estimated DoA from Corollary 1 with the control law dependent on π_k is given by 314.1583, while the area obtained by considering the control law, which is not dependent on π_k, is equal to 2.0612.

- Case 2: SOF Control Design

Now, suppose that only the information about $x_{(1)k}$ is available, given by the measured output y. In this case, it is possible to synthesize an SOF controller from Corollary 2. By using the proposed approach, the largest estimated DoA was obtained for $\mathcal{X} := \{x_k \in \mathbb{R}^2 : |x_{(1)k}| \le 1.09 \text{ and } |x_{(2)k}| \le 0.97\}$.

Fig. 4 presents the ellipsoidal region that represents the estimated DoA and some trajectories initiating inside it, which converge to origin over time. In this case, the area of the estimated DoA is 2.8639. Note that the measured output presents information about the system's nonlinearity. Thus, from the SOF controller designed by applying the proposed methodology, it was possible to obtain a larger estimated DoA than that found using a SF controller which do not take into account the system's nonlinear behavior in the control law.

In Fig. 5, it is possible to verify the relation between the value of the scalar ϵ and the minimum value of the objective function trace(P) obtained from Corollary 2.

One can see that the improvement achieved is not a monotonically increasing function of ϵ. The better result obtained was trace$(P) = 2.5492$, with $\epsilon = 0.1020$.

This case shows the effectiveness of the proposed method when the complete information about system states is not available. Besides that, it illustrates how this new approach can be used to design SOF controllers when the measurement output presents polynomial functions with respect to (x_k).

Example 4. Consider the following nonlinear system, without time-varying parameters, borrowed from Guerra and Vermeiren [15]:

$$\begin{aligned} x_{(1)k+1} &= x_{(1)k} - x_{(1)k}x_{(2)k} + (5 + x_{(1)k})u_k, \\ x_{(2)k+1} &= -x_{(1)k} - 0.5x_{(2)k} + 2x_{(1)k}u_k, \end{aligned} \tag{30}$$

which can be recast in a DAR, such that

$$\begin{aligned} &\pi_k = \begin{bmatrix} x_{(1)k}x_{(2)k} & x_{(1)k}u_k \end{bmatrix}^T, \\ &A_1 = \begin{bmatrix} 1 & 0 \\ -1 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \\ &\Omega_1 = \begin{bmatrix} 0 & x_{(1)k} \\ 0 & 0 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} 0 \\ x_{(1)k} \end{bmatrix}. \end{aligned} \tag{31}$$

Considering $|x_{(1)k}| \le b$, the goal is to obtain the maximum variation for the value b such that there still exists a feasible solution, that is, there is a state-feedback control guaranteeing the

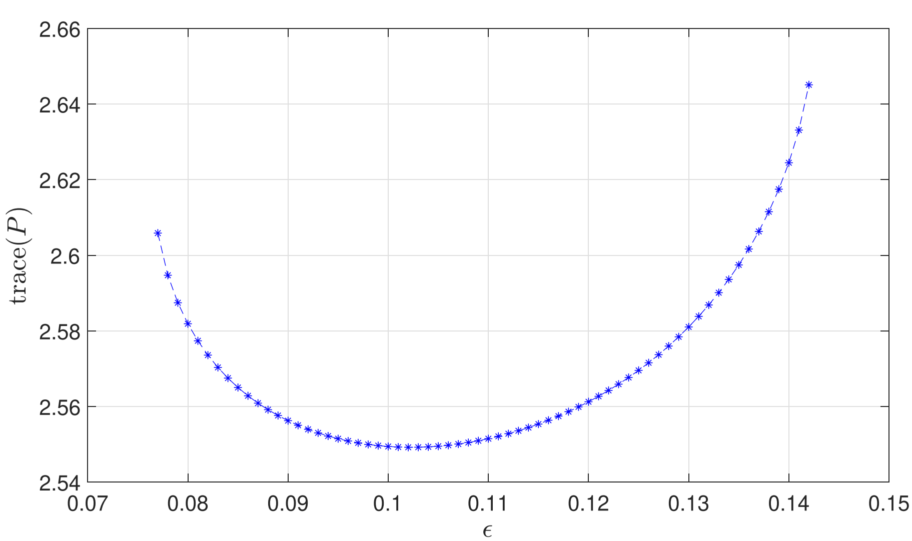

Fig. 5. Relation between ϵ and trace(P).

Table 3
Maximum variations of parameter b obtained using existing conditions and the proposed approach.

Synthesis condition	Maximum b
Theorem 5 in Guerra and Vermeiren [15]	1.539
Theorem 2 in Lendek et al. [19]	1.757
Theorem 4 in Coutinho et al. [11]	2.041
Corollary 3	**4.800**

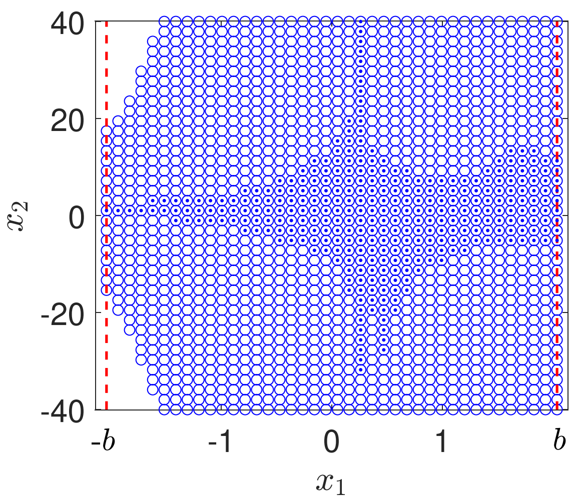

Fig. 6. Stability regions for the closed-loop system with the control gains designed using Corollary 3 (∘) and Theorem 4 in Coutinho et al. [11] (·) with $b = 2.041$.

asymptotic stability. Table 3 presents the largest b obtained from Corollary 3 and other control design conditions existing in the literature in the context of fuzzy discrete-time Takagi-Sugeno fuzzy models [22].

Initially, in Guerra and Vermeiren [15], feasible solutions were obtained for $b \leq 1.539$. This result is improved using the methodology proposed in Lendek et al. [19], which allowed to solve the problem for $b \leq 1.757$. Recently, a better result was obtained in Coutinho et al. [11] by delayed nonquadratic Lyapunov functions, with $b \leq 2.041$. The comparison shows that, from the proposed methodology, it is possible to obtain feasible results for a largest value of b, given by $b = 4.800$, with $\epsilon = 1 \times 10^5$.

It is worth emphasizing that, in the other methodologies presented in Table 3, the regional stabilization with an estimated DoA was not took into account. In this sense, Fig. 6 shows a comparison between the stability regions for the closed-loop system with the control gains designed using the proposed approach and that proposed by [11], considering $b = 2.041$. Note that the proposed conditions provide the largest stability region encompassing that obtained the methodology in Coutinho et al. [11].

Fig. 7 presents state trajectories and the control input sequence for $b = 4.800$. Notice that for this value of b, no feasible solution is found for the other stabilization conditions as described in Table 3.

The control gains obtained for $b = 4.800$ using Corollary 3 are:

$$G_1 = 18.6161, \quad G_2 = 1.8040, \quad G_3 = 18.6133, \quad G_4 = 1.8018.$$

$$\bar{K}_1 = \begin{bmatrix} -1.7773 & -10.1322 \end{bmatrix}, \quad \bar{K}_2 = \begin{bmatrix} -1.3793 & 10.1322 \end{bmatrix},$$
$$\bar{K}_3 = \begin{bmatrix} -2.5799 & -10.1322 \end{bmatrix}, \quad \bar{K}_4 = \begin{bmatrix} -2.1837 & 10.1322 \end{bmatrix}.$$

$$\hat{K}_1 = \begin{bmatrix} 0.1423 & -3.7850 \end{bmatrix}, \quad \hat{K}_2 = \begin{bmatrix} -0.2566 & -3.3862 \end{bmatrix},$$
$$\hat{K}_3 = \begin{bmatrix} 0.1384 & -3.7844 \end{bmatrix}, \quad \hat{K}_4 = \begin{bmatrix} -0.2602 & -3.3857 \end{bmatrix}.$$

From these results, it is possible to conclude that incorporating the vector of nonlinearities into the control law can provide considerably less conservative results in comparison with controllers that use only the information on the system state vector. Furthermore, it is worth registering that we notice that the use of linear annihilators also contributed significantly to reduce conservativeness.

7. Conclusion

This paper has proposed new conditions to compute gain-scheduled SF and SOF controllers with a DoA estimation for rational nonlinear discrete-time systems subject to time-varying parameters. The method builds up on DARs of the system dynamics and parameter-dependent Lyapunov functions, leading to convex optimization problems in terms of LMIs with a linear search parameter. Regarding SF control, a novel condition was presented where the system's nonlinearities are taken into account in the control law, showing favorable results compared to other approaches in the literature. Furthermore, a new solution for SOF control, not

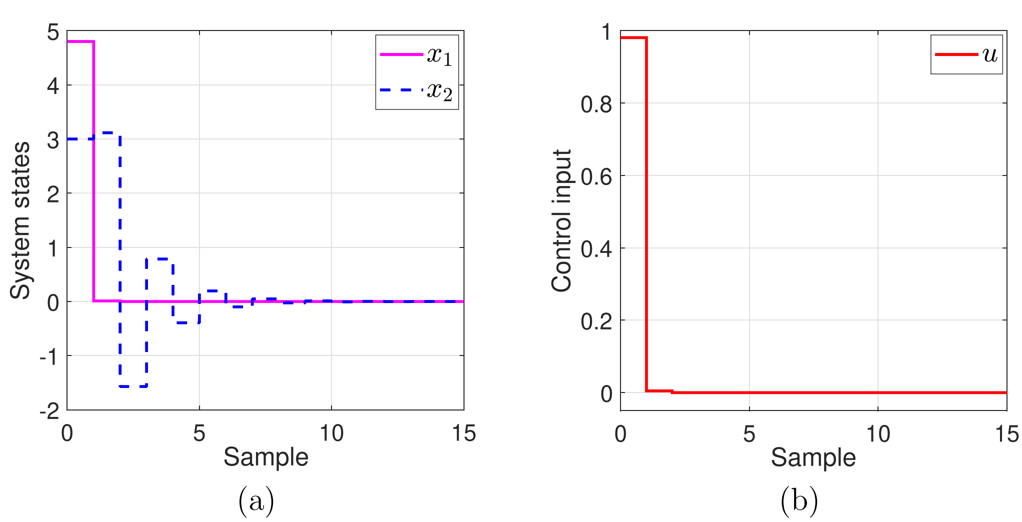

Fig. 7. Time series of the states trajectories and the control input sequence of the closed-loop system with the controller designed using Corollary 3 for $b = 4.800$.

explored in the context of DARs for discrete-time systems, was provided. The proposed SOF control design condition requires neither matrix-rank constraints nor iterative algorithms. In addition, it can be applied to rational nonlinear systems with nonlinear and/or parameter-dependent output matrix. Three numerical examples illustrated the effectiveness and advantages of the proposed method. For future research, we are particularly interested in exploring the use of polynomial Lyapunov functions aiming to improve our results.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Linear annihilator

The matrix $\aleph_x(x_k) : \mathbb{R}^{n_x} \to \mathbb{R}^{n_q \times n_x}$ is a linear annihilator of state vector x_k if $\aleph_x(x_k)x_k = 0$ and $\aleph_x(x_k)$ is linear with respect to x_k. Note that there is no single annihilator to a given system. In this paper we will use the following annihilator proposed by Trofino and Dezuo [45], that takes into account all possible product pairs $x_{(i)k}x_{(j)k}$, $\forall i, j \in \mathcal{I}_{n_x}$ and $i \neq j$:

$$\aleph_x(x_k) = \begin{bmatrix} \Phi_1(x_k) & \Theta_1(x_k) \\ \vdots & \vdots \\ \Phi_{n_x-1}(x_k) & \Theta_{n_x-1}(x_k) \end{bmatrix}, \text{ with} \tag{A.1}$$

$$\begin{cases} \Theta_i(x_k) = -x_{(i)k} I_{n_x-i}, & i \in \mathcal{I}_{n_x-1}, \\ \Phi_1(x_k) = \begin{bmatrix} x_{(2)k} & \dots & x_{(n_x)k} \end{bmatrix}^T, \\ \Phi_i(x_k) = \begin{bmatrix} & x_{(i+1)k} \\ 0_{(n_x-i)\times(i-1)} & \vdots \\ & x_{(n_x)k} \end{bmatrix}, & i \in [2, n_x - 1], \end{cases} \tag{A.2}$$

and the number of rows $n_q = \sum_{j=1}^{n_x-1} j$.

Appendix B. LMI relaxation

In Theorems 1 and 2, it is supposed that inequalities (10) and (17) are dependent on $(x_k, \delta_k, \delta_{k+1})$, Eq. (12) is dependent on (x_k, δ_k), and inequalities (11), (18), and (19) are dependent on (δ_k). Thus, the sufficient conditions provided by these inequalities are modeled through a multi-simplex framework, i.e., the feasibility problem is of infinite dimension, which is not computationally tractable, and some underlying structure should be imposed on the associated matrices to numerically solve the problem. In this sense, a finite set of LMIs in terms of the vertices of the polytopes $\mathcal{X}$ and Δ can be obtained, as follows.

Lemma 2. *Suppose Ψ^n_{ijlm}, with $i, j \in \mathcal{I}_{N_x}$ and $l, m, n \in \mathcal{I}_{N_\delta}$, are matrices of appropriate dimensions, such that*

$$\Psi(x_k, \delta_k, \delta_{k+1}) = \sum_{i=1}^{N_x}\sum_{j=1}^{N_x}\sum_{l=1}^{N_\delta}\sum_{m=1}^{N_\delta}\sum_{n=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}\alpha_{\delta_{(n)k+1}}\Psi^n_{ijlm} < 0. \tag{B.1}$$

If the following LMIs hold for all $i, j \in \mathcal{I}_{N_x}$ and $l, m, n \in \mathcal{I}_{N_\delta}$

$$\begin{aligned} &\Psi^n_{iill} < 0, \quad i = j,\ l = m, \\ &\Psi^n_{ijll} + \Psi^n_{jill} < 0, \quad i < j,\ l = m, \\ &\Psi^n_{iilm} + \Psi^n_{iiml} < 0, \quad i = j,\ l < m, \\ &\Psi^n_{ijlm} + \Psi^n_{ijml} + \Psi^n_{jilm} + \Psi^n_{jiml} < 0, \quad i < j,\ l < m, \end{aligned} \tag{B.2}$$

then inequality (B.1) is satisfied.

Proof. The hypothetical matrix in (B.1) can be rewritten as

$$\begin{aligned} \Psi(x_k, \delta_k, \delta_{k+1}) &= \sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta} \alpha^2_{x_{(i)k}}\alpha^2_{\delta_{(l)k}}\alpha_{\delta_{(n)k+1}}\Psi^n_{iill} \\ &+ \sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x-1}\sum_{j=i+1}^{N_x}\sum_{l=1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha^2_{\delta_{(l)k}}\alpha_{\delta_{(n)k+1}}\left(\Psi^n_{ijll} + \Psi^n_{jill}\right) \\ &+ \sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x}\sum_{l=1}^{N_\delta-1}\sum_{m=l+1}^{N_\delta} \alpha^2_{x_{(i)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}\alpha_{\delta_{(n)k+1}}\left(\Psi^n_{iilm} + \Psi^n_{iiml}\right) \\ &+ \sum_{n=1}^{N_\delta}\sum_{i=1}^{N_x-1}\sum_{j=i+1}^{N_x}\sum_{l=1}^{N_\delta-1}\sum_{m=l+1}^{N_\delta} \alpha_{x_{(i)k}}\alpha_{x_{(j)k}}\alpha_{\delta_{(l)k}}\alpha_{\delta_{(m)k}}\alpha_{\delta_{(n)k+1}}\left(\Psi^n_{ijlm} + \Psi^n_{ijml} \right. \\ &\left. + \Psi^n_{jilm} + \Psi^n_{jiml}\right). \end{aligned}$$

Since $\alpha_{p_{(v)k}} \geq 0$, if (B.2) are satisfied, then condition (B.1) is guaranteed.

Example 5. Let consider a system with $N_x = 2$ and $N_\delta = 2$. By Lemma 2, condition (B.1) is ensured if

$$\begin{aligned} &\Psi^1_{1111} < 0,\ \Psi^1_{1122} < 0,\ \Psi^1_{2211} < 0,\ \Psi^1_{2222} < 0, \\ &\Psi^2_{1111} < 0,\ \Psi^2_{1122} < 0,\ \Psi^2_{2211} < 0,\ \Psi^2_{2222} < 0, \\ &\Psi^1_{1211} + \Psi^1_{2111} < 0,\ \Psi^1_{1222} + \Psi^1_{2122} < 0,\ \Psi^2_{1211} \\ &+ \Psi^2_{2111} < 0,\ \Psi^2_{1222} + \Psi^2_{2122} < 0, \\ &\Psi^1_{1112} + \Psi^1_{1121} < 0,\ \Psi^1_{2212} + \Psi^1_{2221} < 0,\ \Psi^2_{1112} \\ &+ \Psi^2_{1121} < 0,\ \Psi^2_{2212} + \Psi^2_{2221} < 0, \\ &\Psi^1_{1212} + \Psi^1_{2112} + \Psi^1_{1221} + \Psi^1_{2121} < 0,\ \Psi^2_{1212} \\ &+ \Psi^2_{2112} + \Psi^2_{1221} + \Psi^2_{2121} < 0, \end{aligned}$$

resulting in 18 LMIs to be solved.

Lemma 2 can be used to treat computationally inequalities (10) and (17). Finally, considering the polytopic description of matrices $P(\delta_k)$, $G(x_k, \delta_k)$, and $F(\delta_k)$ in (11) and (18), (12), and (19) it is possible to obtain the finite-dimensional LMIs, respectively

$$\begin{bmatrix} 1 & \star \\ a_p & P_l \end{bmatrix} \geq 0, \quad p \in \mathcal{I}_{n_e},\ l \in \mathcal{I}_{N_\delta}, \tag{B.3}$$

$$G^T_{il} + G_{il} > 0, \quad i \in \mathcal{I}_{N_x},\ l \in \mathcal{I}_{N_\delta}, \tag{B.4}$$

$$F^T_l + F_l > 0, \quad l \in \mathcal{I}_{N_\delta}. \tag{B.5}$$

□

It is worth mentioning that the proposed methodology to obtain the LMI conditions can be easily adapted for robust controller design, discussed in Section 5. In this scenario, considering the proposed approach for SF control, we have the gain matrices $G(x_k) = \sum_{i=1}^{N_x} \alpha_{x_{(i)k}} G_i$ and $K(x_k) = \sum_{i=1}^{N_x} \alpha_{x_{(i)k}} K_i$. On the other hand, for the SOF control design, we only have to consider F and H as constant matrices. Notice that, inequalities (10) and (17) continuo to be dependent on $(x_k, \delta_k, \delta_{k+1})$, and Lemma 2 can be applied in a straightforward way for a control law that does not depend on (δ_k).

Appendix C. List of control gains in Examples 1 and 2

The SOF controller gains for Example 1 are given by

$$F_1 = 9.8265, \quad F_2 = 11.8209, \quad H_1 = 11.2750, \quad H_2 = 5.6487.$$

Thus, we have the gain matrices

$$F(\delta_k) = \sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} F_l \text{ and } H(\delta_k) = \sum_{l=1}^{N_\delta} \alpha_{\delta_{(l)k}} H_l, \; N_\delta = 2.$$

The normalized vector α_{δ_k} can be obtained using the following polytopic decomposition

$$\alpha_{\delta_k} = \begin{bmatrix} \beta_{\delta_1(1)k} & \beta_{\delta_1(2)k} \end{bmatrix}^T$$

with

$$\beta_{\delta_1(1)k} = \frac{\overline{\delta}_{(1)k} - \delta_{(1)k}}{\overline{\delta}_{(1)k} - \underline{\delta}_{(1)k}}, \quad \beta_{\delta_1(2)k} = \frac{\delta_{(1)k} - \underline{\delta}_{(1)k}}{\overline{\delta}_{(1)k} - \underline{\delta}_{(1)k}}.$$

where $\overline{\delta}_{(1)k}$ and $\underline{\delta}_{(1)k}$ are the maximum and minimum values of $\delta_{(1)k}$, respectively.

For Example 2, the SF controller gains obtained using Corollary 1 are:

$$G_1 = 0.0377, \quad G_2 = 0.0297, \quad G_3 = 0.0375, \quad G_4 = 0.0300.$$

$$K_1 = \begin{bmatrix} -20.6417 & -50.4402 & -15.1305 & -51.6344 & 0 & 0 & 0 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -17.3959 & -56.6913 & 2.2497 & 57.8815 & 0 & 0 & 0 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} -19.9731 & -57.0337 & -8.0197 & -55.2980 & 0 & 0 & 0 \end{bmatrix},$$
$$K_4 = \begin{bmatrix} -16.9967 & -50.4726 & -4.6552 & 54.0447 & 0 & 0 & 0 \end{bmatrix}.$$

The gain matrices $G(x_k)$ and $K(x_k)$ are given by

$$G(x_k) = \sum_{i=1}^{N_x} \alpha_{x_{(i)k}} G_i \text{ and } K(x_k) = \sum_{i=1}^{N_x} \alpha_{x_{(i)k}} K_i, \; N_x = 4.$$

Similar to the previous case, the normalized vector α_{x_k} can be obtained using the following polytopic decomposition

$$\alpha_{x_k} = \begin{bmatrix} \beta_{x_1(1)k}\beta_{x_2(1)k} & \beta_{x_1(2)k}\beta_{x_2(1)k} & \beta_{x_1(1)k}\beta_{x_2(2)k} & \beta_{x_1(2)k}\beta_{x_2(2)k} \end{bmatrix}^T$$

with

$$\beta_{x_s(1)k} = \frac{\overline{x}_{(s)k} - x_{(s)k}}{\overline{x}_{(s)k} - \underline{x}_{(s)k}}, \quad \beta_{x_s(2)k} = \frac{x_{(s)k} - \underline{x}_{(s)k}}{\overline{x}_{(s)k} - \underline{x}_{(s)k}}, \quad s \in \mathcal{I}_{n_x}.$$

where $\bar{x}_{(s)k}$ and $\underline{x}_{(s)k}$ are the maximum and minimum values of $x_{(s)k}$, respectively.

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